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DYNAMICS
AND HYDROSTATICS.

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AN ELEMENTARY TEXT-BOOK

OF

DYNAMICS AND HYDROSTATICS

BY

R. H. PINKERTON, B.A.(Oxon.)

Author of "Elementary Text-book of Trigonometry" &c. &c.
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BY R. H. PINKERTON, B.A.(OXON.)

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PREFACE.

The design of this Text-book is to explain all that is usually included in a first course of Dynamics. In accordance with the now common acceptation of the term, I have called Dynamics the Science of Force. The book therefore treats of what was formerly, and occasionally still is, called *Theoretical Mechanics.*

Great pains have been taken with the early chapters, which deal with the fundamental principles of Dynamics and with the subject of units. Throughout the book every important proposition is followed by numerical examples fully worked out, and by others which are left as exercises for the student. These examples have been taken from examination papers set in the South Kensington, Civil Service, and University Examinations. Specimen Examination Papers are given at the end of the book.

R. H. PINKERTON.

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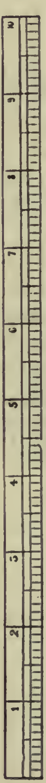
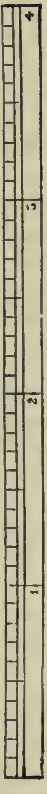
FRENCH AND ENGLISH MEASURES.


A DECIMETRE DIVIDED INTO CENTIMETRES AND MILLIMETRES.


INCHES AND TENTHS.

1 millimetre = ·03937 inch, or about $\frac{1}{25}$ inch.
1 centimetre = ·3937 inch.
1 decimetre = 3·937 inches.
1 metre = 39·37 inches = 3·281 feet.

1 sq. centimetre = ·155 sq. in.

1 cub. centimetre = ·061025 cub. in.

1 gramme = 15·432 grain.
1 kilogramme = 2·205 lbs. avoir.

1 gramme per sq. centim. = 2·0481 lbs. per sq. ft.

1 kilogrammetre = 7·2331 foot-pounds.

1 inch = 2·54 centimetres.
1 foot = 30·48 centimetres.

1 sq. inch = 6·45 sq. centimetres.
1 sq. foot = 929 sq. centimetres.

1 cub. inch = 16·39 cub. centimetres.
1 cub. foot = 28316 cub. centimetres.

1 grain = ·0648 gramme.
1 oz. avoir. = 28·35 grammes.
1 lb. avoir. = 453·6 grammes.

TABLE OF RELATIVE DENSITIES.

LIQUIDS.

Pure water at 4° C.,	1·000
Sea water, ordinary,	1·026
Alcohol, pure,	·791
,, proof spirit,	·916
Ether,	·716
Mercury at 0° C.,	13·596

SOLIDS.

Brass,	7·8 to 8·6
Copper,	8·6 to 8·9
Gold,	19 to 19·6
Iron, cast,	6·9 to 7·3
,, wrought,	7·6 to 7·8
Lead,	11·4
Platinum,	21 to 22
Silver,	10·5
Steel,	7·8 to 7·9
Tin,	7·3 to 7·5
Zinc,	6·8 to 7·2
Ice,	·92
Basalt,	3·0
Brick,	2·0 to 2·2
Brickwork,	1·8
Chalk,	1·8 to 2·8
Clay,	1·9
Glass, crown,	2·5
,, flint,	3·0
Quartz (rock crystal),	2·65
Sand,	1·4
Fir, spruce,	·48 to ·7
Oak, European,	·69 to ·99
Lignum vitæ,	·65 to 1·33
Sulphur, octahedral,	2·05
,, prismatic,	1·98

GASES, at 0° C., and a pressure of 76 centimetres of mercury (one atmosphere).

Air, dry,	·0012932
Oxygen,	·0014298
Nitrogen,	·0012562
Hydrogen,	·00008957
Carbonic acid,	·0019774

TEXT-BOOK OF DYNAMICS.

CHAPTER I.—DEFINITIONS.

1. Dynamics is that branch of Natural Philosophy which treats of the action of force upon matter.

2. Matter is what can be acted on by force.

Bodies in Dynamics are portions of matter.

The quantity of matter in a body is called the **Mass** of the body.

3. Solids, Liquids, and Gases.

We recognize three states in which bodies may exist: (1) the solid state, (2) the liquid state, (3) the gaseous state.

Liquids and gases differ from solids mainly in being unable to sustain the slightest pressure. Gases differ from liquids in their tendency to indefinite expansion. Thus solids have a definite shape and a definite size, liquids have a definite size only, and gases have neither a definite shape nor a definite size.

Many bodies can exist in all three states. Of these a common example is seen in the substance which exists in the solid state as ice, in the liquid state as water, and in the gaseous state as steam.

4. Force.

Force is that which produces or tends to produce change of motion in the matter upon which it acts.

The ideas of force and matter are correlative. We cannot conceive of force except as acting upon matter.

The words "change of motion" in the above definition of force are to be taken in their widest meaning. Thus there is change of motion if the rate of motion of the body is increased or diminished, if the body is brought from a state of rest to a state of motion or from a state of motion to a state of rest, or if the direction in which the body is moving is changed.

As an illustration of the action of force, consider the effect produced on the bullet by the firing of a rifle. Before the gun is fired the bullet is lying at rest, and immediately after the explosion it is moving with considerable velocity. Here we have a case in which a quantity of matter, the bullet, has its state of rest changed into a state of motion. There has, therefore, been a change of motion in the bullet, and, according to our definition of force, what has caused this change is a force. The force is the pressure of the gases produced by the explosion acting on the bullet during the passage of the latter along the barrel of the gun.

Again, take the case of a railway train moving round a curve in the form of a quadrant of a circle at a uniform rate of speed. When the train has moved over the curve there has been no change in the rate of the motion of the train, but there has certainly been a change in the direction of motion. It is clear that, during the passage of the train over the curve, the direction of motion has changed through a right angle. Hence there has been a change in the motion of the train, and this change must be due to some force. The force in this case is the pressure of the outer rail on the wheels of the carriages and engine of the train.

5. Units of Force.

It is common in everyday life to talk of a force of so many pounds or so many tons. Thus the force or pressure of steam in a boiler is generally indicated by a number on the steam-gauge, that number denoting the number of pounds pressure on every square inch of the surface of the boiler.

We see, then, that a force is commonly recognized as a quantity which can be measured, and that the number representing the magnitude of the force is the number of times the force contains some standard or unit force. In measuring the force of steam in a boiler the standard or unit force is the force which is equal to the weight of one pound.

Now, there is this objection on scientific grounds to the use of the weight of one pound as the unit of force, that the weight of one pound of matter is not invariable. A given quantity of matter weighs more at Glasgow, and less at London, than at Manchester. In general the nearer the equator the less is the weight of a given mass. The reason of the difference of weights of the same mass at different points on the earth's surface will be explained in a later chapter. For the present we merely wish to point out that the weight of one pound of matter is not an invariable quantity, and cannot therefore be used as a unit of force in strictly accurate calculations.

Before proceeding to explain how an invariable unit of force may be arrived at, it will be necessary to consider some cases of the motion of bodies without reference to the forces which cause the motion. The science of motion, treated apart from the forces which cause the motion, is called **Kinematics.** Properly speaking, Kinematics is a branch of Pure Mathematics.

6. We may classify the different parts of our subject as follows:—

KINEMATICS
(The Science of Motion).

DYNAMICS.
Including:—
1. STATICS.
   (Which treats of solid bodies at rest under the action of forces.)
2. KINETICS.
   (Which treats of the action of forces in producing motion in solid bodies.)
3. HYDROSTATICS.
   (Which treats of liquids and gases at rest under the action of forces.)

CHAPTER II.—KINEMATICS.

SECTION I.—VELOCITY.

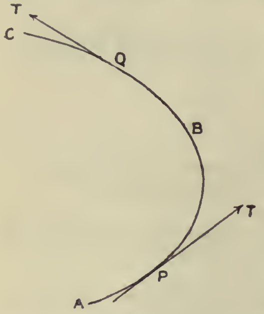
7. Motion, Path, Direction of Motion.

A body is in motion when it is changing its position.

In most of the cases of motion considered in Elementary Dynamics the size of the body need not be taken into consideration, and we can thus look upon the body as a mathematical point having position but not magnitude.

When a body is in motion the straight line or curve in which it is moving is called its path or line of motion.

When the path is a straight line, the direction of motion is, of course, the straight line; but when the path is a curved line, the direction of motion is continually changing. The direction of motion at a given point of the curve is the direction of the tangent to the curve at that point. Thus, if a body is moving in the curve ABC, the direction of motion when the body is at the point P is along PT, the tangent to the curve at the point P. Similarly at the point Q the direction of motion is along QT′, the tangent at Q; so that while the body is moving in the curve from P to Q, the direction of motion changes from the direction of PT to the direction of QT′.



8. Velocity.

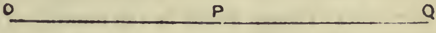
The velocity of a body is its rate of motion.

Velocity may be **uniform** or **variable**.

9. Uniform Velocity.

A body is moving with uniform velocity when it passes over equal spaces, however small, in equal times.

The significance of the words "however small" is illustrated by the following case of motion:—A railway train starts from a station O, passes through a station P at the end of the first hour, and through a station Q at the end of the second hour. If OP = PQ = 30 miles, we have a case of motion in which a body passes over the equal spaces OP and PQ in equal times, and yet the velocity is not necessarily uniform. Between the stations O and P the train may be at rest at one time, and at another time may be moving at the rate of 70 miles an hour; and similar variations are possible between P and Q. It is evident, then, that the velocity between O and Q is not necessarily uniform. If, however, in passing over the distance OQ, the train passes over 44 feet in every second, the limits of possible variation are narrowed, and the velocity must be approximately, if not exactly, uniform. If we knew, further, that in every tenth part of a second it passed over 4·4 feet, we could say that the velocity must be very nearly uniform. If, again, we knew that in every hundredth part of a second it passed over ·44 feet, if in every thousandth part of a second over ·044 feet, if in every ten-thousandth part of a second over ·0044 feet, &c., we should be able to assert that the velocity must be more and more nearly uniform. By such reasoning as this we are led to the idea of a uniform velocity, a velocity in which equal spaces are described in equal infinitesimal intervals of time.



10. A Uniform Velocity is a Measurable Quantity.

The measure of a velocity is the number of times that velocity contains the standard or unit velocity. The standard or unit velocity is the velocity with which a body passes over the unit of space in the unit of time. From this it is evident that the magnitude of the unit velocity, and therefore the measure of any other velocity, will depend on the units of space and time.

11. Units in Actual Use.

There are two systems of units in actual use among scientific men. In one system, the British system, the foot is taken as the unit of length and the second as the unit of time. The unit velocity, therefore, on the British system is the velocity of one foot per second. This velocity would be represented by the number 1, and the number indicating the magnitude of any other velocity would be the number of times that velocity contains the unit velocity, or, what is the same thing, would be the number of feet passed over in one second of time.

In the second system, the French or C.G.S. system, the unit of length is one centimetre* and the unit of time one second. Thus the unit velocity on the C.G.S. system is the velocity of one centimetre per second, and the measure of any other velocity is the number of centimetres passed over in one second.

To compare two velocities we must express them in the same units of space and time, and then compare the numbers measuring them. Thus a velocity of 44 feet per second is half a velocity of 88 feet per second, and double a velocity of 22 feet per second. The velocity of a feet per second is $\frac{a}{b}$ times the velocity of b feet per second.

Example I.—Find the measure of the following velocities when a foot and a second are the units of space and time:—

(1.) 45 miles per hour.
(2.) 15 knots per hour. [Knot = 6080 feet.]
(3.) a yards per b minutes.

(1.) A velocity of 45 miles per hour is equivalent to
,, $45 \times 1760 \times 3$ feet per hour ,,
,, $45 \times 1760 \times 3$ feet per 60×60 secs. ,,
,, $\frac{45 \times 1760 \times 3}{60 \times 60}$ feet per second ,,
,, 66 feet per second.

(2.) A velocity of 15 knots per hour is equivalent to
,, $\frac{15 \times 6080}{60 \times 60}$ feet per second ,,
,, $25\frac{1}{3}$ feet per second.

---

* Centimetre = $\frac{1}{10}$ decimetre = $\frac{1}{100}$ metre. A metre is equivalent to 39·37079 inches, so that the centimetre, or the hundredth part of the metre, is approximately four-tenths of an inch.

(3.) A velocity of a yards per b minutes is equivalent to
„ $\frac{3a}{60b}$ foot per second „
„ $\frac{a}{20b}$ foot per second.

Example II.—Express the following velocities in terms of the centimetre and second:—

(1.) 30 metres per hour.
(2.) 15 decimetres per minute.
(3.) a centimetres per b days (24 hours).

(1.) Since metre = 100 centimetres,
a velocity of 30 metres per hour is equivalent to
„ 30×100 centimetres per 60×60 secs. „
„ $\frac{30 \times 100}{60 \times 60}$ centimetres per second „
„ $\frac{5}{6}$ centimetre per second.

(2.) Since decimetre = 10 centimetres,
a velocity of 15 decimetres per minute is equivalent to
„ 15×10 centimetres per 60 seconds „
„ $\frac{15 \times 10}{60}$ centimetres per second „
„ $2\frac{1}{2}$ centimetres per second.

(3.) A velocity of a centimetres per b days is equivalent to
„ a centimetres per $b \times 24 \times 60 \times 60$ secs. „
„ $\frac{a}{86400\,b}$ centimetre per second.

EXAMPLES I.

1. Find the measures of the following velocities when a second is the unit of time and a foot the unit of space:—

(1.) 3 yards per second.
(2.) $18\frac{1}{2}$ inches per minute.
(3.) 400 miles per hour.
(4.) 3 fathoms per second.
(5.) 110 furlongs per hour.
(6.) $45\frac{1}{3}$ yards per hour.
(7.) 600 miles per 10 hours.
(8.) 55 feet per $3\frac{1}{4}$ minutes.
(9.) a yards per second.
(10.) a inches per b seconds.
(11.) a miles per b hours.
(12.) $(a^2 - b^2)$ furs. per $(a+b)^2$ min.

2. Which is the greatest of the following velocities?

(1.) 1740 yards per minute.
(2.) 86 feet per second.
(3.) 59 miles per hour.

3. The measure of a certain velocity is 10 when a foot and a second are the units of space and time. Find the measure of the same velocity when the units of space and time are:—

(1.) A yard and a second.
(2.) A yard and a minute.
(3.) A furlong and an hour.
(4.) A fathom and a day (24 hours).
(5.) An inch and a year (365 days).
(6.) 93 million miles and 17 min.

4. Find the measures of the following velocities when a centimetre is the unit of space and a second the unit of time:—

(1.) 3 metres per second.
(2.) $18\frac{1}{2}$ decimetres per min.
(3.) 400 centimetres per hour.
(4.) 3 decimetres per hour.
(5.) a centimetres per b seconds.
(6.) a metres per b hours.
(7.) $(a-b)^2$ decimetres per (a^2-b^2) minutes.

5. Which is the greatest of the following velocities?

(1.) 950 centimetres per min.
(2.) $1\frac{3}{8}$ decimetres per second.
(3.) 580 metres per hour.

6. Two bodies are moving with uniform velocities, one describing 36 miles in one hour and 20 minutes, the other 55 feet in $1\frac{1}{4}$ seconds. Compare the two velocities.

7. The measure of a certain velocity is v when the units of space and time are a mile and an hour. Find the measures of the same velocity when the units are:—

(1.) A foot and a second.
(2.) A yard and a minute.
(3.) A furlong and a day.

12. Variable Velocity.

A body is said to be moving with **variable** velocity when it does not pass over equal spaces in equal times.

The measure of the velocity *at a particular instant* of a body moving with variable velocity is the number of units of space the body *would* pass over in one unit of time if the velocity were to continue the same as at the moment under consideration for one unit of time. Thus, when we say that at a certain time a ship is sailing at the rate of 8 knots an hour, we do not mean that it will pass over 8 knots in the next hour, because the speed may be increased or diminished during the hour. We mean that if it continues to sail at the same rate as at the moment under consideration, then it will pass over 8 knots in the next hour.

The student will notice that the velocity of a body is a characteristic of the motion of the body at a particular instant of time.

13. Distance passed over in a given Time by a Body moving during that time with given Uniform Velocity.

If a body, moving with a velocity v, passes over s units of space in t units of time, then

$$s=vt.$$

For in one unit of time the body passes over v units of space, in two units of time it passes over $2v$ units of space, in three units of time over $3v$ units of space, and so on. Thus in time t, that is in t units of time, the body will pass over vt units of space. Hence, since s denotes the space passed over in this time, it follows that $s=vt$.

In using this formula the student must be careful to use the same units of time and space throughout.

Example 1.—A train travels at the rate of 66 feet per second; what distance will it travel in $1\frac{1}{2}$ hours?

Taking a foot and a second as the units of space and time, we have $v=66$. The time is $1\frac{1}{2}$ hours, or 90 minutes, or 5400 seconds, and therefore $t=5400$. Hence s, the space described,

$$\begin{aligned}&=vt=66\times 5400 \text{ feet}\\&=67\tfrac{1}{2} \text{ miles.}\end{aligned}$$

Example 2.—A point is moving at the rate of 150 yards a minute. How long will it take to go 10 miles?

Taking a yard as the unit of space and a minute as the unit of time, we have $v=150$, $s=10\times 1760$; hence

$$10\times 1760=150\times t,$$

from which

$$\begin{aligned}t&=\frac{10\times 1760}{150}\\&=117\tfrac{1}{3} \text{ minutes.}\end{aligned}$$

EXAMPLES II.

1. A body is moving with the uniform velocity of 10 feet per second. How far will it go in (1) $3\frac{1}{4}$ hours, (2) $\frac{1}{3}$ of a day, (3) 12 hours, 4 minutes, 5 seconds?
2. A body is moving with the uniform velocity of 15 yards per hour. How long will it take to go (1) $\frac{1}{2}$ mile, (2) 350 yards, (3) 1 mile, 1 furlong, 1 yard?
3. A steam-ship runs a knot in $4\frac{1}{2}$ minutes. How long will it take to go 100 miles? [Knot = 6080 feet.]
4. A carrier-pigeon flies 100 miles in 120 minutes. How long does it take to go 1450 yards?
5. A railway train, 120 yards long, passes over a bridge, 80 feet long, at the rate of 30 miles an hour. How long does the train take to pass completely over the bridge?
6. How long would a train, travelling at the rate of a mile a minute, take to go from the earth to the sun, a distance of 93 millions of miles? Give result in years, days, and hours.
7. Find the velocity of the earth in its orbit round the sun, taking the orbit as a circle of 93 million miles radius described with uniform velocity in $365\frac{1}{4}$ days. Give result in feet and seconds.

14. Average Velocity.

When a body passes over a given space with variable velocity, the velocity with which it would pass uniformly over the same space in the same time is called the **average** velocity over that space. Thus the average velocity of a train travelling in 10 hours from Glasgow to London, a distance of 400 miles, is the uniform velocity with which the train would travel 400 miles in 10 hours, and that velocity is evidently 40 miles per hour.

The formula of the preceding Article evidently applies to average velocity.

EXAMPLE.—From 7.19 p.m. on October 18th to 12 noon on October 19th a steamship runs a distance of 257 knots. Find the average velocity in feet per second.

Here the time is 16 hours 41 minutes or 60060 seconds, and the distance 257 knots or 257×6080 feet. Hence by the formula the average velocity is

$$v = \frac{257 \times 6080}{60060} = 26\frac{50}{3003} \text{ feet per second.}$$

EXAMPLES III.

1. An express train travels from London to Oxford, a distance of 65 miles, in 75 minutes. Find the average velocity of the train in feet per second.
2. An express train travels from Glasgow to Carlisle, a distance of 100 miles, in 2 hours and 35 minutes. Find its average velocity in feet per second.
3. A steam-ship ran in 3 consecutive days distances of 370, 389, and 373 knots respectively. Find its average velocity over the three days in yards per minute.
4. A steam-ship ran from Sandy Hook to Plymouth, a distance of 2951 knots, in 7 days, 17 hours, 5 minutes. What was its average velocity in knots per hour?
5. Find the average velocity of a man who runs 1 mile in 6 minutes, the next in 7 minutes, the next in 8 minutes, and the fourth in 9 minutes.

15. Graphical Representation of Velocities.

A velocity is completely specified when we are given its magnitude and direction. Since straight lines possess the same two characteristics, a velocity may be represented in a drawing by a straight line. To represent a velocity by a straight line, we draw in the direction of the velocity a straight line of any length, and cut off from this straight line a length containing as many units of length as the given velocity contains units of velocity. It is immaterial what unit of length we take to represent in our drawing the unit of velocity, but in drawing lines to represent two or more velocities we must keep to the same unit of length. The result will be a diagram of velocities drawn to scale. Thus if we represent a velocity of 1 foot per second by a line 1 inch in length, we should represent a velocity of 10 feet per second by a line 10 inches in length, a velocity v feet per second by a line v inches in length, and so on.

16. Composition of Velocities.

We have seen that a body in motion has at every point of its path a definite velocity in a definite direction. Now there are cases of motion in which this velocity is compounded or made up of two or more velocities. We have an example of such a case in the motion of a man walking across the deck of a steamer while the steamer is travelling forward. It is evident that the motion of the man *in space* will depend both on his own motion across the deck and on the motion of the ship forward. Hence the actual velocity of the man in space is made up of two velocities, viz. (1) his own velocity relatively to the steamer, and (2) the velocity of the steamer through the water. When, as in this case, the actual velocity of a body is compounded or made up of two or more velocities, each of these velocities is called a **component velocity,** and the actual velocity is called the **resultant velocity.**

17. Resultant of Two or more Velocities in the same Straight Line.

It is evident that if we have two velocities in the same straight line and in the same direction, the resultant velocity will be a velocity in the same straight line and in the same direction, the magnitude being equal to the sum of the magnitudes of the two component velocities. Thus a cannon-ball fired right ahead from a gunboat travelling forwards will have a velocity equal to the sum of the velocity of the gunboat and the velocity due to the explosion of the gun.

In symbols, if v_1 and v_2 denote the two component velocities in the same straight line and in the same direction, and if v denote the resultant velocity, then

$$v = v_1 + v_2.$$

So also the resultant of two component velocities in the same straight line, but in opposite directions, will be the difference between the two components. If, in the above illustration of the cannon-ball fired from the gunboat, the gun were fired right astern, the actual velocity of the ball would be equal to the difference between the velocity due to the explosion of the gun and the velocity of the gunboat through the water.

With the above notation, we have in this case

$$v = v_1 \sim v_2.$$

If we look upon a velocity in one direction as a positive quantity, and a velocity in the opposite direction as a negative quantity, we shall have a simple rule for finding the resultant of any number of component velocities in the same straight line. Take one direction as the positive direction, consider all the components in that direction as positive quantities, and consequently all the components in the opposite direction as negative quantities. Then—

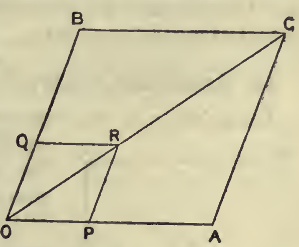
The magnitude of the resultant velocity is the algebraical sum of the components, and the direction of the resultant velocity will be in the positive or negative direction according as this algebraical sum is positive or negative.

Example.—A steamer can steam 12 knots an hour in still water. What will be its velocity in a current which runs at the rate of $2\frac{1}{4}$ knots per hour (1) when running with the current, (2) when running against the current?

$$\begin{aligned}\text{Velocity with the current} &= 12 + 2\tfrac{1}{4}\\ &= 14\tfrac{1}{4} \text{ knots per hour.}\\ \text{Velocity against the current} &= 12 - 2\tfrac{1}{4}\\ &= 9\tfrac{3}{4} \text{ knots per hour.}\end{aligned}$$

18. Parallelogram of Velocities.

When a body has two component velocities not in the same straight line, the resultant velocity is found by the construction indicated in the following Proposition, which is known as the **Parallelogram of Velocities**:—



If a body has two velocities represented in magnitude and direction by two straight lines OA and OB, and if the parallelogram OACB be completed, the resultant velocity shall be represented in magnitude and direction by that diagonal OC of the parallelogram which passes through O.

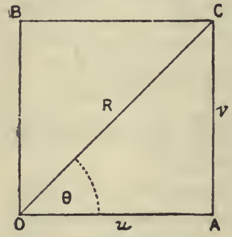
By Article 15 OA and OB represent respectively the spaces passed over with the component velocities in one unit of time, which for clearness we shall take to be one second. In one second, then, the body would with one component velocity pass over the space represented by OA, and with the other component velocity would pass over the space represented by OB. Now it follows from the idea of component velocities that the position of the body at the end of the second will be the same as if it moved through the space represented by OA, while the line OA is being carried through a space equal and parallel to that represented by OB. But AC is equal and parallel to OB. Hence at the end of the second the body will be at C.

Also, if OP and OQ represent respectively the spaces the body would describe with the given velocities in any the same part of a second, it is evident that OP bears to OA the same ratio that OQ bears to OB. Hence if the parallelogram OPRQ be completed, it will be similar to the parallelogram OACB, and therefore R will lie in OC, and OR will be the same part of OC that OP is of OA or OQ of OB. Hence the body moves along OC.

Since, then, the body is at C at the end of the second, and since it moves along the line OC, it follows that OC represents the actual space described in one second. Also, since in any given part of a second the space described OR is the same part of OC that OP is of OA, it follows that the motion along OC is uniform. On the same scale, therefore, on which OA and OB represent the component velocities, OC represents the actual or resultant velocity.

19. Resultant of two Velocities at Right Angles.

Let u and v be the two component velocities in directions at right angles to each other, and let these velocities be represented graphically by the lines OA and OB respectively at right angles to each other. Then by the parallelogram of velocities the resultant velocity is represented in magnitude and direction by the diagonal OC of the rectangle OACB.



Let R denote the resultant velocity, and θ the angle it makes with the velocity u. Then the three lines OA, AC (which is equal to OB), and OC are proportional to u, v, and R. But by Euclid I. 47

$$OC^2 = OA^2 + AC^2;$$

therefore $$R^2 = u^2 + v^2;$$

and taking the sq. root, $R = \sqrt{u^2 + v^2}$. ....(1).

Also $$\tan AOC = \frac{AC}{OA} = \frac{OB}{OA};$$

and therefore $\tan\theta = \dfrac{v}{u}$. ....(2).

Formulæ (1) and (2) completely determine the resultant.

EXAMPLE.—What is the resultant velocity of a body which has a velocity eastwards of 6 miles an hour and a velocity northwards of 12 miles an hour?

Here $u = 6$, $v = 12$, and therefore R, the resultant, is given by the formula

$$R^2 = u^2 + v^2 = 36 + 144 = 180;$$

from which $R = \sqrt{180}$

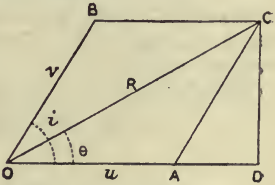
$= 13{\cdot}42$ miles per hour approximately.

Let θ denote the angle the direction of the velocity makes with the east line, then

$$\tan\theta = \frac{v}{u} = \frac{12}{6} = 2.$$

From a table of natural tangents we find that the angle whose tangent is 2 is an angle of 63° 26′. This, therefore, is the angle which the resultant velocity makes with the east line.

20.* Resultant of two Component Velocities inclined at any Angle.



Let u and v be the two component velocities inclined at an angle i, let R be the resultant velocity, and let θ be the angle which the resultant velocity makes with the velocity u.

Let OA and OB represent graphically the component velo-

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[*] See Appendix for method of finding resultant in certain particular cases without trigonometry, and for the method of graphical solution.

cities u and v respectively; the resultant velocity R is represented by OC, the diagonal of the parallelogram of which OA and OB are adjacent sides.

Since $AC = OB$, and angle $OAC = 180° - i$, it follows from trigonometry that

$$\begin{aligned} OC^2 &= OA^2 + AC^2 - 2OA \,.\, AC \cos OAC \\ &= OA^2 + OB^2 - 2OA \,.\, OB \cos(180° - i) \\ &= OA^2 + OB^2 + 2OA \,.\, OB \cos i. \end{aligned}$$

Hence, since OA, OB, and OC are respectively proportional to u, v and R,

$$R^2 = u^2 + v^2 + 2uv \cos i,$$

and $$R = \sqrt{u^2 + v^2 + 2uv \cos i} \quad \text{.................. (1).}$$

Let CD be drawn perpendicular on OA, produced if necessary. Then $AC = OB$, the angle $CAD = \text{angle } AOB = i$,

$$\begin{aligned} OD &= OA + AD = OA + AC \cos i \\ &= OA + OB \cos i, \end{aligned}$$

and $$CD = AC \sin DAC = OB \sin i.$$

Hence $$\tan\theta = \tan AOC = \frac{CD}{OD} = \frac{OB \sin i}{OA + OB \cos i}.$$

$$= \frac{v \sin i}{u + v \cos i} \quad \text{.................................... (2).}$$

Formulæ (1) and (2) completely determine the resultant velocity.

If $i = 90°$, $\sin i = 1$ and $\cos i = 0$, and the formulæ reduce to those of the preceding article.

Example.—A body has two component velocities 4 and 10 inclined at an angle of 60°. Find the magnitude and direction of the resultant velocity.

Taking $u = 4$, $v = 10$, $i = 60°$, we get

$$\begin{aligned} R &= \sqrt{16 + 100 + 2 \,.\, 4 \,.\, 10 \,.\, \tfrac{1}{2}} = \sqrt{156} \\ &= 12{\cdot}49. \end{aligned}$$

$$\tan\theta = \frac{10 \sin 60°}{4 + 10 \cos 60°} = \frac{10 \frac{\sqrt{3}}{2}}{4 + 10 \,.\, \frac{1}{2}} = \frac{5\sqrt{3}}{9}$$

$$= {\cdot}962 = \tan 43°\ 54'.$$

The magnitude of the resultant is 12·49, and the direction makes an angle of 43° 54′ with the smaller component.

EXAMPLES IV.

In this set of Examples the following natural tangents are required:—

$\tan 78^\circ\ 35' = 4{\cdot}95.$
$\tan 67^\circ\ 23' = 2{\cdot}4.$
$\tan 61^\circ\ 56' = 1{\cdot}875.$
$\tan 33^\circ\ 41' = {\cdot}\dot{6}.$
$\tan 32^\circ\ 12' = {\cdot}6298.$
$\tan 28^\circ\ 37' = {\cdot}\dot{5}\dot{4}.$
$\tan 16^\circ\ 4' = {\cdot}288.$

1. Find the resultant and the angle it makes with the smaller component in each of the following cases, in which two component velocities are given:—

(1.) 5 and 12 at right angles.
(2.) 8 and 15 at right angles.
(3.) $5\sqrt{3}$ and 5 at right angles.
(4.) 99 and 20 at right angles.
(5.) $4\sqrt{3}$ and 12 at right angles.
(6.) 7 and 8 at 30°.
(7.) 21 and 24 at 60°.
(8.) 15 and $18\sqrt{2}$ at 45°.

2. Find the magnitude of the resultant of the two component velocities in each of the following cases:—

(1.) $a+b$ and $a-b$ at right angles.
(2.) $(m+n)^2+2(m+n)$ and $2(m+n+1)$ at right angles.
(3.) $m+n\sqrt{3}$ and $m-n\sqrt{3}$ at right angles.
(4.) $m\sqrt{2}+n\sqrt{3}$ and $m\sqrt{2}-n\sqrt{3}$ at right angles.
(5.) $1+m\sqrt{2}$ and $1-m\sqrt{2}$ at 120°.

3. What is the actual velocity in magnitude and direction of a ship which is sailing 3 knots an hour eastwards, while it is drifting with a current 2 knots an hour southwards?

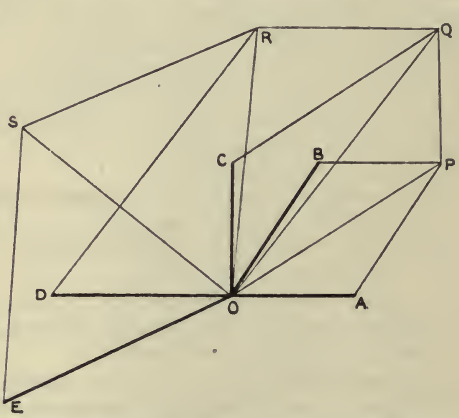
4. What is the actual velocity of a grain of sand which is falling vertically through water at the rate of 6 feet per second, while it is being carried horizontally by a current at the rate of $1\frac{1}{2}$ feet per second?

5. A gunboat is steaming at the rate of 15 knots an hour, and a shot is fired with a velocity of 1000 feet per second in a direction making an angle of 60° with the direction of motion of the ship. What is the actual velocity of the shot in feet per second? [Knot = 6080 feet.]

21. Resultant of any number of Component Velocities. Graphical Method.

The resultant of any number of component velocities may be found graphically by repeated application of the parallelogram of velocities.

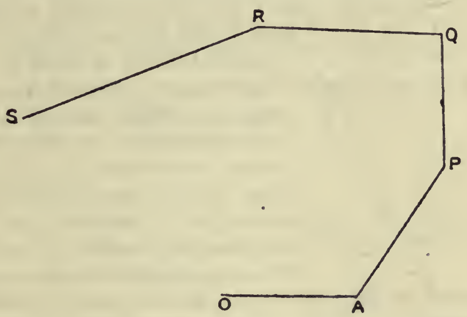
Suppose that there are five component velocities represented by the lines OA, OB, OC, OD, OE. By the parallelogram of velocities the resultant of the two velocities represented by OA and OB is a velocity represented by the diagonal OP of the parallelogram constructed on OA and OB as adjacent sides. Replace the velocities represented by OA and OB by the velocity represented by OP, and compound this resultant velocity with the velocity represented by OC. The resultant will be a velocity represented by OQ, which will therefore be equivalent to the three velocities represented by OA, OB, OC. Compound the velocity represented by OQ with the velocity represented by OD, and the resultant will be a velocity represented by OR. This velocity compounded with the velocity represented by OE will give as the resultant a velocity represented by OS. Thus OS represents the velocity which is the resultant of the five velocities represented by OA, OB, OC, OD, OE.



Since AP is equal and parallel to OB, PQ equal and parallel to OC, QR equal and parallel to OD, and RS equal and parallel to OE, it is evident that the above construction could be replaced by the following simpler construction:—

Draw OA; from A draw AP equal and parallel to OB;

from P draw PQ equal and parallel to OC; from Q draw QR equal and parallel to OD; and from R draw RS equal and parallel to OE. Then the resultant of the five velocities is represented by the line joining O to the point S so found.



We should proceed in a similar way whatever be the number of component velocities.

It is evident that if S coincides with O, the resultant will be zero. Hence we have the following theorem, known as the **Polygon of Velocities**:—

If a body has component velocities represented in magnitude and direction by the sides of a closed polygon taken in order, then the resultant velocity is zero, or the body is at rest.

A particular case of the polygon of velocities, in which the polygon is a triangle, is known as the **Triangle of Velocities.**

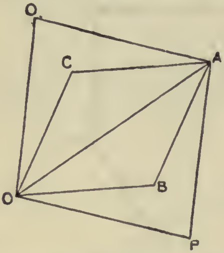
The magnitude and direction of the resultant of any number of component velocities could be found algebraically by repeated application of the formulæ of Article 20, but that mode of procedure would be found very laborious. A simpler method will be given in Article 24.

22. Resolution of a Velocity.

When we are given a velocity we can find two other velocities to which the given velocity shall be equivalent.

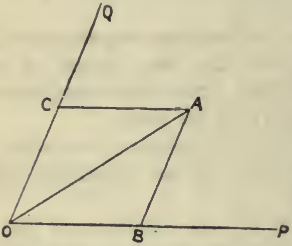
This process, known as the resolution of a velocity into two components, is therefore the converse of the composition of two velocities into one resultant velocity.

We can resolve a velocity into two components in an infinite number of ways. Let OA represent a velocity; then we can form an infinite number of parallelograms of which OA shall be the diagonal. Two such parallelograms are OBAC and OPAQ. By the parallelogram of velocities the velocity represented by OA is equivalent to the two velocities represented by OB and OC; also to the two velocities represented by OP and OQ, and generally to the velocities represented by the sides of any parallelogram of which OA is a diagonal.



23. Resolution of a Velocity into two Components in two given directions.

Let OA represent any velocity, and let OP and OQ be two given directions. Required to resolve the given velocity into two components along OP and OQ respectively. From A draw AB and AC respectively parallel to OQ and OP; then by the parallelogram of velocities the velocity represented by OA is equivalent to the two velocities represented by OB and OC. Hence the components along OP and OQ are represented respectively by OB and OC.



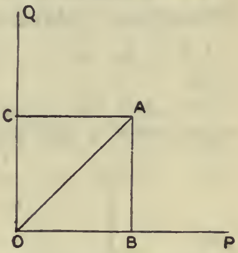
The most important case of the resolution of a velocity is the case in which the two given directions are at right angles to each other.

If OA represent any velocity, then OB and OC represent respectively the components along the two lines OP and OQ at right angles to each other. Let u be the measure of the velocity represented by OA, and θ the angle AOB which OA makes with OP. Then since

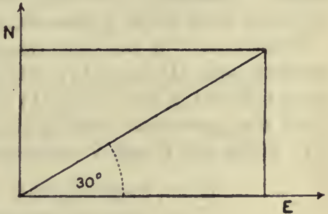
$$OB = OA \cos AOB = OA \cos\theta,$$
$$OC = OA \cos AOC = OA \sin\theta,$$

it follows that the component along OP is $u \cos\theta$, and that the component along OQ is $u \sin\theta$.

From this Article it is evident that a velocity can be resolved into two components along two given directions in only one way.



We shall often speak of the component of a velocity in a given direction. By this is meant the component which, compounded with the component in the direction at right angles to the given direction, would give a resultant equal to the given velocity. We have seen above that the component in a given direction is found by multiplying the given velocity by the cosine of the angle between the given direction and the direction of the given velocity. In the last figure the component along OP is found by multiplying u, the given velocity, by the cosine of the angle AOP, and the component in the direction OQ is found by multiplying u by the sine of the angle AOP, that is, by the cosine of the angle AOQ.



Example.—A train is travelling at the rate of 40 miles an hour in a direction E. 30° N. How fast is it moving eastwards, and how fast northwards?

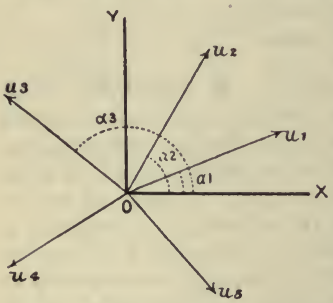
We resolve the given velocity into two components, one eastwards and the other northwards. Since the direction of the given velocity makes an angle of 30° with the east line, and an angle of 60° with the north line,

$$\begin{aligned}\text{Eastward component} &= 40\cos 30^\circ = 40\frac{\sqrt{3}}{2}\\ &= 34{\cdot}64 \text{ miles per hour.}\\ \text{Northward component} &= 40\times\cos 60^\circ = 40\times\tfrac{1}{2}\\ &= 20 \text{ miles per hour.}\end{aligned}$$

Hence the train is travelling eastwards at the rate of 34·64 miles per hour, and northwards at the rate of 20 miles per hour.

24. Resultant of any number of Component Velocities. Algebraical Method.

Let $u_1, u_2, u_3, \ldots\ldots\ldots$ be any number of component velocities.



Take any two straight lines through O, OX and OY, at right angles to each other. Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots$ be the angles which $u_1, u_2, u_3, \ldots\ldots\ldots$ make respectively with the line OX, the angles being all measured in the same direction. Thus in the figure the angle which u_4 makes with OX is the angle measured from OX to u_4 in the direction contrary to the direction of the hands of a watch. This is an angle greater than two right angles. Similarly, the angle which u_5 in the figure makes with OX is an angle greater than three right angles.

Resolve each of the velocities $u_1, u_2, u_3, \ldots\ldots\ldots$ into its components along OX and OY. The components of u_1 are $u_1\cos\alpha_1$ along OX and $u_1 \sin\alpha_1$ along OY; and similarly for $u_2, u_3, \ldots\ldots\ldots$ Find the algebraical sums of the components along OX and OY, which call U and V respectively. Then

$$U = u_1 \cos\alpha_1 + u_2 \cos\alpha_2 + u_3 \cos\alpha_3 + \ldots\ldots\ldots$$
$$V = u_1 \sin\alpha_1 + u_2 \sin\alpha_2 + u_3 \sin\alpha_3 + \ldots\ldots\ldots$$

The given system of velocities $u_1, u_2, u_3, \ldots\ldots\ldots$ is thus equiválent to the two velocities, U along OX and V along OY.

Let R denote the resultant velocity and θ the angle it makes with OX, then by Article 19,

$$R = \sqrt{U^2 + V^2} \dots\dots\dots\dots(1).$$

$$\tan\theta = \frac{V}{U} \dots\dots\dots\dots(2).$$

Formulæ (1) and (2) determine the magnitude and direction of the resultant velocity.

It is generally convenient to choose the direction of one of the given velocities as the direction of one of the lines OX, OY. If two of the given velocities are at right angles, it will evidently simplify the working to choose the directions of these two velocities as the directions of OX and OY.

Example.—A particle receives simultaneously 3 velocities, viz. 60 feet per second north, 88 feet per second W. 30° S., and 60 feet per second E. 30° S. Find the magnitude and direction of the resultant velocity.

We resolve northwards and westwards.

(1.) Take the velocity 60 N.

Northward component $= 60$.

Westward component $= 0$.

(2.) Take the velocity 88 W. 30° S.

$$\begin{aligned}\text{Northward component} &= 88\cos(90^\circ + 30)\\ &= 88\cos 120^\circ\\ &= -44\\ \text{Westward component} &= 88\cos 30^\circ = 88\frac{\sqrt{3}}{2}\\ &= 44\sqrt{3}.\end{aligned}$$

(3.) Take the velocity 60 E. 30° S.

$$\begin{aligned}\text{Northward component} &= 60\cos 120^\circ = 60(-\tfrac{1}{2})\\ &= -30\\ \text{Westward component} &= 60\cos 150^\circ = 60\left(-\frac{\sqrt{3}}{2}\right)\\ &= -30\sqrt{3}.\end{aligned}$$

The sum of the northward components is

$$60 - 44 - 30 = -14.$$

This, being negative, indicates a southward component.

The sum of the westward components is

$$44\sqrt{3} - 30\sqrt{3} = 14\sqrt{3}.$$

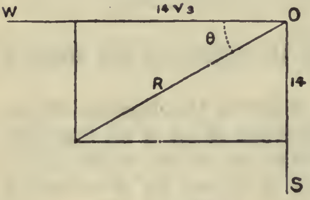
Hence the 3 given velocities are equivalent to the two velocities, 14 southwards and $14\sqrt{3}$ westwards. Hence if R denote the resultant and θ the angle it makes with the west line,

$$R^2 = (14)^2 + (14\sqrt{3})^2,$$

from which $R = 28$.

Also $\tan\theta = \frac{14}{14\sqrt{3}} = \frac{1}{\sqrt{3}}$,

which gives $\theta = 30°$.



Hence the resultant is 28 feet per second W. 30° S.

EXAMPLES V.

[The student will find it a useful exercise to work out Examples of the composition and resolution of velocities as well by the graphical method as by the algebraical method. Let him draw the velocities to scale, taking off the angles by means of a protractor; then, by measuring the length of the line which in his drawing represents the required velocity, an approximate value of the magnitude of that velocity can immediately be found.]

1. Resolve the velocity 10 into two components at right angles, when the angle between the direction of the given velocity and the direction of one of the components is

(1.) 30°; (2.) 45°; (3.) 60°; (4.) 120°; (5.) 150°.

2. A velocity 6 is resolved into two components at right angles, one of the components being $3\sqrt{3}$. What is the other? What are the angles the given velocity makes with the two components?

3. A velocity u is resolved into two components at right angles. In each of the following cases one component is given; find the other component and the angle u makes with this latter:—

(1.) $\frac{u\sqrt{3}}{2}$. (2.) $\frac{u}{2}$. (3.) $\frac{u\sqrt{2}}{2}$.

4. A train is moving up a gradient of 1 in 100 at the rate of 15 miles an hour. How fast is it moving horizontally, and at what rate is it rising vertically?

5. A body has 4 component velocities, viz. 60 feet per second east,

70 feet per second north, 80 feet per second west, 90 feet per second south. Find the magnitude of the resultant velocity.

6. A body has 6 component velocities, 6, 5, 4, 3, 2, 1, taken in order, the angle between any adjacent two being 60°. Find the magnitude and direction of the resultant velocity.

7. A body has simultaneously given to it three velocities, viz. 60 feet per second E. 30° N., 70 feet per second W. 30° N., and 80 feet per second S. Find the magnitude of the resultant velocity.

8. An abandoned vessel is drifting north-eastwards at the rate of 10 knots a day. If a knot corresponds to a difference of latitude of $1'$, how long will the vessel take to drift through 10° of latitude?

9. Two trains moving on parallel rails, the distance between which may be neglected, at the uniform rates of 30 miles per hour and 45 miles per hour, are 5 miles apart at a certain instant. How far apart will they be at the end of 6 minutes from that instant (1) if they are moving in the same direction; (2) if they are moving in opposite directions?

10. Two trains start from a junction, and travel with equal uniform velocities along two lines of rails inclined to each other at an angle of 60°. Show that the distance between the trains is increasing uniformly, and at a rate equal to the velocity of either train.

11. Show that the resultant of two equal velocities u and u, along lines inclined to each other at an angle α, is

$$2\,u \cos \tfrac{1}{2}\alpha.$$

12. A velocity u is resolved into two components in two directions, making angles α and β with the direction of the given velocity. Show that the two components are respectively

$$\frac{u \sin\beta}{\sin(\alpha+\beta)} \quad \text{and} \quad \frac{u \sin\alpha}{\sin(\alpha+\beta)}.$$

Verify this by showing that these two component velocities in directions inclined at an angle $(\alpha+\beta)$ have a resultant equal to u.

SECTION II.—ACCELERATION.

25. Acceleration.

Acceleration is rate of change in the magnitude of a velocity.

If a body is moving in a straight line with uniform velocity, there is no change either in the magnitude or the direction of the velocity, and there is therefore no acceleration.

If a body is moving in a straight line with increasing velocity, the rate of increase in the velocity at any particular instant is called the acceleration at that instant. It may be either uniform or variable.

If a body is moving in a straight line with decreasing velocity, the rate of decrease in the velocity at any particular instant is called the **retardation** at that instant. By using the algebraical terms positive and negative, with their corresponding symbols + and −, to represent difference of direction, we may speak of a retardation as a negative acceleration; that is, an acceleration in the direction opposite to that which is considered positive.

Lastly, if a body is moving in a curve the direction of velocity is continually changing, so that there is at every instant an acceleration. In this chapter we shall consider only cases of acceleration of the velocity of a body moving in a straight line.

26. Measure of an Acceleration.

A uniform acceleration is measured by the number of units of velocity by which the velocity is increased or diminished in one unit of time.

Thus if we take a foot as the unit of space, and a second as the unit of time, the unit velocity will be a velocity of one foot per second, and the unit of acceleration will be the acceleration by which the velocity is increased one foot per second in one second. This is expressed shortly by saying that the unit acceleration is the acceleration of one foot per second per second.

Any other acceleration will be measured by the velocity in feet per second added on in one second. Thus with the same units of space and time an acceleration 18 means an acceleration by which the velocity of the body is increased in one second by the velocity 18 feet per second. We express this shortly by saying that the acceleration 18 is an acceleration of 18 feet per second per second.

The measure of a variable acceleration[1] at any particular instant is the number of units of velocity by which the velocity would be changed in one unit of time if the acceleration were to remain the same as at the moment contemplated for one unit of time.

It is evident that, as in the case of a velocity, the measure of an acceleration will depend on the units of space and time that are adopted.

Example 1.—A body is found to be moving in a straight line at the rate of 5 feet per second; a quarter of a minute afterwards at the rate of 50 feet per second; and half a minute afterwards at the rate of 95 feet per second. Show that this motion is consistent with the velocity of the body being uniformly accelerated, and find the measure of the acceleration in feet and seconds.

In a quarter of a minute or 15 seconds the velocity has increased by 45 feet per second. If this increase of velocity were uniform, the increase in one second would be a velocity of 3 feet per second. Again, in half a minute or 30 seconds the velocity has increased by 90 feet per second, and, if this increase were uniform, this would result from an increase of 3 feet per second in one second. Hence the motion throughout is consistent with the velocity being increased at the rate of 3 feet per second in every second; that is, the motion is consistent with a uniform acceleration of 3 feet per second per second.

Example 2.—An acceleration is denoted by 30 when a foot and a second are the units of space and time. What will be the measure of the same acceleration when the units of space and time are a yard and a minute?

An acceleration of 30, when the units of space and time are a foot and a second, means a velocity of 30 feet per second added on in one second. But a velocity of 30 feet per second is equivalent to a velocity of 10 yards per second, or a velocity of 600 yards per minute. Hence the given acceleration is equivalent to a velocity of 600 yards per minute added on in one second; that is, equivalent to a velocity of 600×60 yards per minute added on in one minute, or a velocity of 36,000 yards per minute added on in one minute. Thus the given acceleration is represented in the new units by the number 36,000.

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[1] Compare the measure of a variable velocity, Art. 12.

EXAMPLES VI.

1. A body moving with uniform acceleration has a velocity of 9 feet per second at a certain instant, and 15 minutes afterwards it has a velocity of 18 yards per second. What is the acceleration in feet and seconds?

2. A body whose velocity is known to be uniformly accelerated is moving at the rate of 100 yards per minute, and 10 seconds afterwards at the rate of 160 yards per minute. What is the acceleration in feet and seconds?

3. A train starts from rest and at the end of one minute has a velocity of 15 miles per hour. Thirty seconds later its velocity is 33 feet per second. Show that this motion is consistent with uniform acceleration.

4. A train moving with uniform acceleration increases its speed from 5 miles an hour to 20 miles an hour during an interval of 2 minutes. What is the acceleration in feet and seconds?

5. In a minute a train moving with uniform acceleration increases its velocity from 44 feet per second to 51 miles per hour. What is the measure of the acceleration in feet and hours?

6. A train whose velocity is uniformly retarded is moving at the rate of 15 miles an hour. Eleven seconds ago its velocity was 11 yards per second. What was its velocity, in feet and seconds, (1) 7 seconds ago, (2) 17 seconds ago? What will be its velocity (3) after 3 seconds, (4) after 11 seconds?

7. 32 is the measure of an acceleration when a foot and a second are the units of space and time. What would be the measure of the same acceleration when the units of space and time are (1) a yard and a minute; (2) half a foot and half a second; (3) a mile and an hour; (4) a furlong and a day (24 hours)?

8. Express an acceleration of 300 centimetres per second per second in terms of (1) a metre[1] and an hour; (2) a decimetre[2] and a day (24 hours); (3) a kilometre[3] and a minute.

9. What is the number representing an acceleration of a feet per second per second when the units of space and time are (1) b feet and one second; (2) one foot and c seconds; (3) b feet and c seconds?

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[1] Metre = 100 centimetres. [2] Decimetre = 10 centimetres.

[3] Kilometre = 1000 metres = 100,000 centimetres.

10. If the acceleration of a feet per second per second is b times the acceleration of c yards per second per second, show that

$$a = 3bc.$$

27. Body moving in a Straight Line with Uniform Acceleration.

A body is moving with uniform acceleration a in a straight line. If u be the velocity of the body at a certain instant of time, to find v the velocity t units of time after that instant, and s the space passed over during that time t.

Since the body is moving with uniform acceleration a, it follows that in every unit of time the velocity is increased by a units of velocity, and therefore in t units of time the velocity u is increased by $a\,t$ units of velocity. Hence, v denoting the velocity at the end of the time t, we have

$$v = u + at \text{..............................(1).}$$

Hence the velocity increases uniformly with the time, and it follows that the average velocity during the time t will be half the sum of the velocities at the beginning and end of that time. Hence

$$\begin{aligned}\text{Average velocity} &= \tfrac{1}{2}(u + u + at)\\ &= u + \tfrac{1}{2}at.\end{aligned}$$

The space s described during the time t will (Article 14) be found by multiplying the average velocity by the time; thus

$$\begin{aligned}s &= t(u + \tfrac{1}{2}at)\\ &= ut + \tfrac{1}{2}at^2 \text{..............................(2).}\end{aligned}$$

From formulæ (1) and (2) another formula can be obtained by algebra, thus:—Squaring the expression for v in formula (1), we get

$$\begin{aligned}v^2 &= u^2 + 2uat + a^2t^2\\ &= u^2 + 2a(ut + \tfrac{1}{2}at^2)\\ &= u^2 + 2as \text{..............................(3),}\end{aligned}$$

a formula connecting the velocity with the space passed over.

Special Cases—

I. Body starting from rest.

In this case $u = 0$, and the formulæ (1), (2), and (3) become

$$v = at, \quad s = \tfrac{1}{2}at^2, \quad v^2 = 2as.$$

II. When the body starts with a velocity u in one direction and has an acceleration a in the opposite direction.

To adapt the formulæ to this case we consider the acceleration as negative, and change a into $-a$. The formulæ then become

$$v=u-at$$
$$s=ut-\tfrac{1}{2}at^2$$
$$v^2=u^2-2as.$$

Example 1.—A body with the acceleration 10 feet per second per second begins to move with a velocity of $7\tfrac{1}{2}$ feet per second. After what time would its velocity be quadrupled, and what space would it describe in that time?

Taking a foot and a second as the units of space and time, we have $a=10$, $u=7\tfrac{1}{2}$, $v=4\times 7\tfrac{1}{2}=30$, and it is required to find s and t.

By formula (1),

$$30=7\tfrac{1}{2}+10t,$$

from which $t=2\tfrac{1}{4}$ seconds.

By formula (2),

$$\begin{aligned}s&=7\tfrac{1}{2}\times 2\tfrac{1}{4}+\tfrac{1}{2}\times 10\times(2\tfrac{1}{4})^2\\&=42{\cdot}1875 \text{ feet.}\end{aligned}$$

Hence the time and the space described have been found.

Example 2.—A body starting from rest moves with an acceleration of 20 feet per second per second. What space does it describe in 6 seconds, and what is its velocity at the end of that time?

Here $u=0$, and we use the formulæ of the first special case to find v and s. The acceleration $a=20$, and $t=6$. Hence

$$v=20\times 6=120 \text{ feet per second,}$$

and $$s=\tfrac{1}{2}\times 20\times 6^2=360 \text{ feet.}$$

Example 3.—A train is travelling at the rate of 60 miles an hour when the steam is shut off. If in 1 minute the speed is reduced uniformly to 30 miles per hour, what is the retardation in feet and seconds? What distance does the train travel in this time?

We have here a case of retardation or negative acceleration, and we therefore make use of the formulæ of the second special case.

Velocity of 60 miles per hour = velocity of 88 feet per second.
„ 30 „ = „ 44 „
1 minute = 60 seconds;

hence, when a foot and a second are the units of space and time,

$$u=88,\ v=44,\ t=60;$$

therefore, by the first formula,

$$44 = 88 - 60a,$$

from which $a = \frac{11}{15}$.

Hence the retardation is $\frac{11}{15}$ feet per second per second.

To find the distance the train travels in the given time we use the third formula; thus,

$$44^2 = 88^2 - 2 \times \tfrac{11}{15} \times s,$$

from which $\frac{22}{15}s = 88^2 - 44^2,$

which gives $s = 3960$ feet,

$= 1320$ yards, the required distance.

EXAMPLES VII.

1. A body moves from rest with an acceleration of 8 feet per second per second. What space does it describe in $\frac{1}{2}$ minute?
2. A train, moving with uniform acceleration, takes 5 minutes to travel over the first mile. What velocity has it acquired at the end of that time? How long does it take to travel over the second mile?
3. The velocity of a body changes from 50 feet per second to 60 feet per second while the body is passing over 11 feet. What is the acceleration (supposed uniform) in feet and seconds?
4. A body starts from rest and moves with an acceleration of 10 feet per second per second. What spaces does it describe in the first, second, third, fourth, and fifth seconds respectively of its motion?
5. A body describes distances of 120 yards, 228 yards, and 336 yards in successive tenths of a second. Show that this is consistent with uniform acceleration, and find the measure of the acceleration if a yard and a minute are the units of space and time.
6. A body, moving with uniform acceleration, begins to move from a state of rest. It is observed to move through 55 feet in a certain 2 seconds, and through 77 feet in the next 2 seconds. What distance does it describe in the first six seconds of its motion?
7. A body, uniformly accelerated, starts from rest and goes over 5 inches in the first second. How far would it go in the first two, first three, first four, and first five seconds respectively?
8. A train, moving at the rate of 20 miles per hour, is brought to

rest in 100 seconds, the retardation being uniform. Find the space passed over meanwhile.

9. A body, uniformly accelerated, starts from rest and passes over a certain space in the first second. Show that it will pass over 15 times that space in the next three seconds.

10. A body, uniformly accelerated, starts from rest and passes over b feet in p seconds. Find how long it will take to pass over the next b feet.

11. The speed of a railway train increases uniformly for 10 minutes from the time of starting from rest; it then remains constant for 15 minutes; and lastly it decreases at a uniform rate for five minutes, but twice as rapidly as it increased in the first 10 minutes. Compare the distance travelled in each of the three periods with the total distance.

12. A particle A starts from a given point and moves with uniform acceleration. Afterwards another body B starts from the same point with a greater acceleration. Show that difference of velocities when B overtakes A : A's velocity : : A's velocity when B started : the velocity acquired by A in the remainder of the motion.

13. A ball, rolling down a straight groove, passes in succession four points, A, B, C, D, whose distances apart are, A to B, 37·05 centimetres; B to C, 101·2 centimetres; C to D, 157·55 centimetres. The time occupied by the ball in going from A to B is 1·3 seconds; in going from B to C, 2·2 seconds; in going from C to D, 2·3 seconds.

(1.) Show that the motion of the ball is consistent with uniform acceleration.

(2.) Find the value of the acceleration, the centimetre and second being units of space and time.

(3.) The velocities of the ball when passing the points A and D.

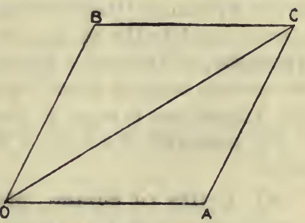
28. Accelerations are Compounded and Resolved like Velocities.

Accelerations have two characteristics, magnitude and direction, and therefore, like velocities, they may be represented graphically by straight lines.

If a body has two accelerations in directions inclined at any angle, and if these accelerations are represented in magnitude and direction by two straight lines, then the line representing the resultant or actual acceleration is found by the construction of Article 18, which we shall refer to as the **Parallelogram Construction.** Corresponding, then, to the parallelogram of velocities we have the following theorem, known as the **Parallelogram of Accelerations**:—

If a body has two accelerations represented in magnitude and direction by the two straight lines OA and OB, the resultant acceleration is represented by the diagonal OC of the parallelogram constructed on OA and OB as adjacent sides.

This proposition follows immediately from the parallelogram of velocities. The lines OA and OB represent accelerations, that is, the velocities added on in the directions OA and OB respectively in the unit of time. Hence, by the parallelogram of velocities, the diagonal OC represents the actual velocity added on in the unit of time, that is, OC represents the actual or resultant acceleration.



Since accelerations are compounded by the parallelogram construction, all the consequences of the parallelogram of velocities given in Articles 19 to 24 will be true also when we substitute the word acceleration for the word velocity. Thus the algebraical formulæ for finding the resultant of two or more velocities will still obtain if we suppose the symbols used to denote the measures of velocities to stand for the measures of acceleration. In elementary dynamics it is seldom required to compound accelerations, but we shall meet with cases of motion in which it will be necessary to find the component of an acceleration in a given direction. If a denote the acceleration and i the angle the direction of the acceleration makes with the given direction, then, as in the case of a velocity, the component in the given direction will be $a \cos i$.

CHAPTER III.—PRINCIPLES OF DYNAMICS.

29. Mass. Density.

We have defined (Article 2) the mass of a body as the quantity of matter in the body. In measuring the mass of a body we select a certain mass as our standard or unit, and the number of times the mass we are measuring contains the standard or unit will be the measure of that mass.

Two units of mass are in use among scientific men, the **pound** and the **gramme.** The pound is the unit mass in the British system of units, and is defined to be a mass equal to the mass of a certain lump of metal deposited in the Office of Exchequer at London. The gramme is the unit of mass in the French or C.G.S. system of units, and is defined to be a mass equal to the thousandth part of the mass of a **kilogramme**, the mass of a kilogramme being equal to the mass of a certain lump of metal deposited in the Archives at Paris. The kilogramme is equal to about 2⅕ pounds.

The **density** of a body is the mass in unit volume of the body.

30. Units of Space, Mass, and Time.

We have now stated (Articles 11 and 29) the units of space, mass, and time—the three fundamental units of Dynamics—adopted in the British system of units and in the French or C.G.S. system. We may tabulate these units thus:—

	British System.	French or C.G.S. System.
Unit of Space	foot	centimetre.
Unit of Mass	pound	gramme.
Unit of Time	second	second.

The C.G.S. system (so called from the initial letters of the words centimetre, gramme, second) is the system in general use among scientific men on the Continent, and is rapidly replacing the British system in this country.

31. Momentum.

The **momentum** at any instant of a body in motion is the product of the mass of the body and the velocity at that instant.

The unit of momentum in the British system is the momentum of the mass of one pound moving with the velocity of one foot per second. In the C.G.S. system the unit momentum is the momentum of the mass of one gramme moving with a velocity of one centimetre per second.

The momentum of a body of mass m moving with velocity u is mu.

The change of momentum of a moving body in a given time is the difference between the momentum at the end and the momentum at the beginning of that time. Thus if m be the mass of a body moving at the beginning of a certain time with velocity u, and at the end of that time with velocity u', the change of momentum during that time is

$$mu' - mu = m(u' - u).$$

From the second form we see that the change of momentum is equal to the product of the mass of the body and the change of its velocity.

Example 1.—The mass of a cubic foot of a substance A is 62·5 lbs., and the mass of 36 cubic inches of another substance B is 75 oz. Compare the density of A with the density of B.

Since cubic foot $= 12 \times 12 \times 12$ cubic inches, and lb. $= 16$ oz.,

mass of a cubic inch of A $= \dfrac{62 \cdot 5 \times 16}{12 \times 12 \times 12} = \dfrac{125}{216}$ oz.,

also mass of a cubic inch of B $= \dfrac{75}{36} = \dfrac{25}{12}$ oz.

Hence density of A : density of B

$$= \left\{ \begin{array}{l} \text{mass of a cubic inch of A : mass of a} \\ \quad \text{cubic inch of B} \end{array} \right.$$

$$= \frac{125}{216} : \frac{25}{12} = 5 : 18.$$

Example 2.—A body whose mass is 100 lbs., is moving at one instant with a velocity of 30 miles an hour, and 7 seconds afterwards with a velocity of 17 yards per second. What is the total change of momentum? Supposing this change to take place uniformly, what is the change per second?

Velocity of 30 miles per hour = velocity of 44 feet per second,
Velocity of 17 yards per second = velocity of 51 feet per second;

hence change of velocity $= 51 - 44 = 7$ feet per second, and total change of momentum, in British units, is 7×100 or 700.

This change takes place in 7 seconds, and, if the change is uniform, the change in one second is $\frac{1}{7} \times 700$ or 100.

EXAMPLES VIII.

1. Of two bodies one has a volume of 5 cubic inches and the other of one-fifth of a cubic foot. The mass of the first body is 15 oz. and of the second 12·8 lbs. What is the ratio of the mass of the first to the mass of the second? What is the ratio of the density of the first to the density of the second?

2. If a cubic centimetre of water contained exactly a gramme of matter, what would be the quantity of matter, estimated in grammes, in a cubic foot of lead? [Foot = 30·45 centimetres; density of lead = 11·445 times the density of water.]

3. Express in British units the momentum of—

   (1.) A mass of 4 lbs. moving with a velocity of 10 yards per second.

   (2.) A mass of $2\frac{1}{4}$ cwt. moving with a velocity of $\frac{1}{2}$ yard per minute.

   (3.) A train of 200 tons moving with a velocity of 40 miles per hour.

   (4.) A tennis-ball of $\frac{1}{2}$ oz. moving with a velocity of 40 feet per second.

4. Express in C.G.S. units the momentum of—

   (1.) A mass of 4 grammes moving with a velocity of 30 centimetres per hour.

   (2.) A mass of 14 grammes moving with a velocity of 13 metres (100 centimetres) per second.

   (3.) A mass of 15 grammes moving with a velocity of 4 metres per minute.

   (4.) A mass of 20 kilogrammes (1000 grammes) moving with a velocity of 3 metres per second.

5. Find in British units the change of momentum of a mass of $2\frac{1}{2}$ tons which at one time is moving with a velocity of 40 feet per second, and at a subsequent time with a velocity of 40 miles per hour.

   If this change of momentum takes places uniformly in 10 seconds, what is the change in one second?

6. Find in C.G.S. units the change of momentum of a mass of

4 kilogrammes which at one time is moving with a velocity of 800 centimetres per second, and at a subsequent time with a velocity of 10 metres per second.

If this change of momentum takes place uniformly in 5 seconds, what is the change in one second?

7. Of two bodies one has a mass of 2 cwt. and moves at the rate of 4 feet per second; the other has a mass of 21 lbs. and moves at the rate of 120 yards per minute. Find the ratio of the momentum of the former body to that of the latter.

32. Newton's Laws of Motion.

The following three laws, known as Newton's Laws of Motion, regarding the effect of force upon matter, form the foundation of the whole science of Dynamics.

33. Law I.

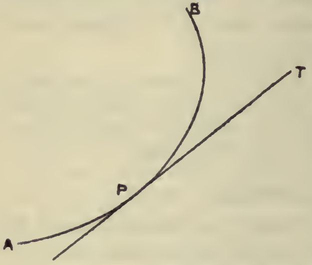
Every body continues in a state of rest or of uniform motion in a straight line except in so far as it is compelled to change that state by force acting on it.

This law asserts, *firstly*, that a body at rest will remain at rest unless it is acted on by some force; *secondly*, that a body moving at any instant with any velocity in any direction will, if not acted on by force after that instant, continue to move in that direction with that velocity.

Thus if a body is at rest at one instant, and is afterwards found to be in motion, there is a change in its state of rest, and this change is due to the action of some force.

Again, if a body is moving in a straight line with a certain velocity, and at a later time is found to be moving, either in another straight line or in the same straight line with a different velocity, there is a change in the state of motion of the body, and this change must be due to the action of some force.

Lastly, if a body is moving in a curve AP, there is a continual change in the motion, and consequently there must be force continually acting on the body. For if, when the body is at the point P in the curve, there are no forces acting on it, the body will continue to move in the direction in which it is moving at that instant with the velocity it has at that instant. Now the direction of motion at any point of a curve is along the tangent to the curve at that point, and, therefore, if there are no forces acting on the body at the point P, the body will continue to move along the tangent PT with the velocity it has at that point. It follows that as the body moves in the curve, it is continually being deflected from the tangent along which it is momentarily moving. There is thus a continual change in the direction of the motion, and this change must be due to the continual action of force.



We may state the first law of motion thus:—

Whatever changes the motion of a body in magnitude or in direction is force.

According to this law, therefore, all bodies have a tendency to preserve their state of rest or of uniform motion in a straight line, that is, all bodies tend to resist change of motion. This tendency of a body to resist change of motion is called **Inertia.** We shall see presently that the inertia of a body is proportional to the mass of the body. It is on account principally of greater inertia that it is more difficult to get up the speed of a heavy goods train than of a light passenger train; and for the same reason a goods train running at the same rate of speed as a passenger train will continue to move longer when the steam is shut off.

34. Law II.

Change of motion of a body is proportional to the force acting on the body, and takes place in the direction of the straight line in which the force acts.

By the words "change of motion" in this law Newton denotes what is now called change of momentum.

The first law of motion asserts that change of momentum is due to the action of force alone, and this law adds that the amount of change is proportional to the magnitude of the force.

It is to be noticed that, in comparing the magnitudes of two forces by comparing the changes of momentum produced by the two forces respectively, we must compare the changes of momentum produced in equal times.

Since the force acting on a body is proportional to *change* of momentum, the effect of a force is the same whether the body is at rest or in motion when the force begins to act. Hence if a constant force acts on a body, the change of momentum produced in the first unit of time will be equal to that produced in the second unit of time, and in general will be equal to that produced in any unit of time during which the force acts. Now the change of momentum during the unit of time is the mass of the body multiplied by the change of velocity in the unit of time, Art. 31, and the change of velocity in the unit of time is the acceleration. We thus arrive at this important result: *A constant force acting on a body causes the body to move with a constant acceleration, and the force is proportional to the product of the mass of the body and the acceleration.*

Thus two forces are equal when they produce equal accelerations in equal masses, and in general forces are proportional to the accelerations they produce in equal masses.

Again, in order that forces may produce equal accelerations in unequal masses, each force must be proportional to the mass on which it acts.

Lastly, if equal forces act on unequal masses, the accelerations produced will be inversely proportional to the masses. Thus the greater the mass of a body, the greater is its tendency to resist acceleration, that is, the greater the mass of a body, the greater is its inertia.

EXAMPLE.—A body A, mass 8 lbs., is moving northwards with a velocity of 30 miles an hour at a certain instant, and at the same time another body B, mass 8 oz., is moving southwards with a velocity of 55 feet per second. At a subsequent time A is found to be moving southwards with a velocity of 33 feet per second, and B to be moving northwards with a velocity of 15 miles per hour. Compare the forces acting on A and B during the interval.

The velocity of A changes from that of 30 miles an hour, or 44 feet per second, northwards to that of 33 feet per second southwards. Hence the change is $44+33=77$ feet per second southwards, and the change of momentum, in British units, is 8×77 southwards.

Also the velocity of B changes from that of 55 feet per second southwards to that of 15 miles per hour northwards, that is, to that of 22 feet per second northwards. Hence the change of velocity is $55+22=77$ feet per second northwards, and the change of momentum, the mass of B being $\frac{1}{2}$ lb., is $\frac{1}{2} \times 77$ northwards.

By the second law of motion the forces producing these changes are proportional to the changes, and therefore

$$\begin{aligned} &\text{force acting on A : force acting on B} \\ &= 8 \times 77 : \tfrac{1}{2} \times 77 = 16 : 1. \end{aligned}$$

Hence the magnitude of force acting on A must be sixteen times the magnitude of the force acting on B. The former acts southwards, the latter northwards.

35. Absolute or Dynamical Units of Force.

We saw in Article 5 that there is this objection to the use of the weight of one pound as the unit of force, that it is not the same at all parts of the earth's surface. An invariable unit of force is suggested by the second law of motion, and is called an absolute or dynamical unit of force. A dynamical unit of force is defined as follows:—

An absolute or dynamical unit of force is that force which, acting for one unit of time on unit mass, produces unit change of velocity.

The magnitude of any particular absolute unit of force will depend, of course, on what units of space, mass, and time are adopted.

Once the absolute unit of force is fixed, the measure of any other force will be the number of units of velocity generated by that force in unit mass during one unit of time; or, what is the same thing, the measure of a force in absolute units is the acceleration it produces in unit mass.

36. Poundal. Dyne.

There are two absolute units of force in actual use in Dynamics, the **poundal** and the **dyne**. The poundal is the absolute unit of force in the British system of units, and the dyne the absolute unit of force in the C.G.S. system.

Thus—

The poundal is that force which, acting on the mass of one pound for one second, generates in that mass a velocity of one foot per second.

Similarly—*The dyne is that force which, acting on the mass of one gramme for one second, generates in that mass a velocity of one centimetre per second.*

37. Formula connecting the Acceleration a, produced in Mass m, by a Force f.

We have seen that a force is measured by the acceleration it causes in unit mass, the unit of force being the force which produces unit acceleration in unit mass. If then a force produces a units of acceleration in unit mass, the force is represented numerically by a; and if it produces a units of acceleration in m units of mass, the force is represented by ma (Article 35). Hence if f denote this force, we have

$$f = ma \text{.........................} (1),$$

from which

$$a = \frac{f}{m} \text{.........................} (2).$$

These two formulæ give the connection between Kinematics and Kinetics. From the first we infer that if a mass m is moving with an acceleration a, the force which causes this motion is, in absolute units, ma. From the second we learn that if a mass m is acted on by a force f, expressed in absolute units, the acceleration which this force produces is $\frac{f}{m}$.

Example 1.—What is the force in poundals that in 11 minutes produces a velocity of a mile a minute in a cwt.?

A velocity of a mile a minute is the same as a velocity of 88 feet per second. This velocity is produced in 11 minutes, and therefore the velocity generated in one second is $88 \div 11 \times 60$, or $\frac{2}{15}$ feet per second. The acceleration therefore in feet and seconds is $\frac{2}{15}$. The mass in which this acceleration is produced is 112 lbs. Hence the force is—

$$112 \times \tfrac{2}{15} = \tfrac{224}{15} = 14{\cdot}9\dot{3} \text{ poundals.}$$

EXAMPLE 2.—What is the force in dynes required to move 69 grammes 500 centimetres from rest in 34·5 seconds?

Let a, expressed in C.G.S. units, be the acceleration which the force produces; then with this acceleration the body passes over 500 centimetres from rest in 34·5 seconds. Hence by Article 27—

$$500 = \tfrac{1}{2} \times a \times (34{\cdot}5)^2,$$

from which $$a = \frac{1000}{(34{\cdot}5)^2};$$

and the required force is $$= \frac{1000}{(34{\cdot}5)^2} \times 69,$$

$$= 57{\cdot}96 \text{ dynes very nearly.}$$

EXAMPLES IX.

1. A mass of 20 lbs. moving with a velocity of 40 feet per second is brought to rest by a constant force acting in the direction opposite to the direction of motion after passing over 400 feet. What is the force in poundals?
2. A constant force acting on a mass of 12 grammes for 10 seconds gives it a velocity of 15 centimetres per second. Express the force in dynes, and calculate the velocity it would give in 5 seconds to a mass of 8 grammes.
3. A constant force acting on a mass of 20 grammes for 3 seconds gives it a velocity of 24 centimetres per second. Find the force in dynes.
4. A force acts on a mass of 8 oz. for 6·9125 minutes, and produces a velocity of 10 feet per second. What is the magnitude of the force in poundals?
5. A particle is moving in a straight line with a velocity of 5 feet per second. A quarter of a minute afterwards it is moving at the rate of 50 feet per second, and after a further ¼ minute at the rate of 95 feet per second. Show that this motion is consistent with the action of a constant force on the body.
6. A ball of 10 lbs. has during an interval of one minute increased its velocity from 10 feet per second to 50 feet per second under the influence of a constant force. What is the force in poundals?
7. What constant horizontal force in poundals is required to stop a train of 100 tons mass running at 50 miles an hour (1) in one minute, (2) in 200 yards?
8. A mass of 81 pounds, moving with an initial velocity of 200 feet

per second, is subject to a uniform retarding force of 108 poundals. Show how far it will move while its velocity falls from 200 to 196 feet per second; also while it falls from 4 feet per second to zero.

9. A carriage whose mass is 3 tons is drawn from rest along a smooth level road by a rope, and acquires a velocity of 10 feet per second in 5 seconds. If the carriage be stopped, and the pull on the rope be trebled, while the load is doubled, what will be the velocity at the end of 10 seconds?

10. A force F, acting on a mass of 5 lbs., increases the velocity of the mass in every second by 12 feet per second. A second force F_1, acting on a mass of 28 lbs., increases the velocity of this mass in every second by 7½ feet per second. Find the ratio of F to F_1.

11. The mass of the carriages of a train is 200 tons, and the carriages are being dragged along with an acceleration of 3 feet per second per second. What is the tension of the coupling between the engine and the first carriage?

12. If the inch, the second, and the ounce be taken as the units of length, time, and mass, compare the unit of force with the poundal.

13. If the metre, the minute, and the kilogramme were taken as the units of length, time, and mass, compare the unit of force with the dyne.

38. Force of Gravity.

It is a matter of common observation that all bodies let drop fall towards the surface of the earth. From this we infer that the earth exerts a certain attractive force on all bodies. This attractive force is called the **force of gravity**, and the amount of this force on any given mass is called the **weight** of that mass.

In this connection the student must carefully distinguish between the mass of one pound and the weight of one pound. The *mass* of one pound is the *quantity of matter* equal to a certain standard quantity (Article 29), and is quite independent of force. The *weight* of one pound is the *force* with which the mass of one pound is attracted to the earth's surface. It is usual to speak of a force of so many pounds or of so many tons. Such expressions must be understood to mean the weight of so many pounds or so many tons. When the word pound is used, the student must gather from the context whether it is the mass of a pound or the force equal to the weight of a pound that is meant.

If two bodies, such as a feather and a coin, be let fall together, it is found that they do not reach the ground at the same time, the coin falling in a much shorter time than the feather. By experiment we learn that this is not due to the difference in shape, size, or material of the feather and the coin, but is due solely to the presence of the atmosphere. The experiment by means of which this is proved is known under the name of the "guinea and feather experiment." In this experiment a guinea and a feather are let drop together from the top of a vertical glass tube, from which the air has been withdrawn by means of an air-pump. It is found that the guinea and the feather fall vertically side by side, and reach the bottom of the tube at the same time.

What is true of the guinea and feather is true, of course, of other bodies also; so that, if the atmosphere were removed, any number of bodies, whatever be their masses, volumes, and densities, let fall together from the same height, would all fall vertically side by side, and reach any horizontal plane at the same moment. From this we infer that the bodies must be falling at every instant with the same velocity, and consequently with the same acceleration. We have thus a case of different bodies moving with the same acceleration, which, by Art. 34, requires that the force acting on each body shall be proportional to the mass of that body. Now, the force acting on each body in this case is the weight of each body, so that the "guinea and feather experiment" lead us to the conclusion that, at the same place on the earth's surface, the weights of bodies are proportional to their masses.

It is usual in dynamics to denote the acceleration caused by gravity in the motion of a body falling *in vacuo* by g, the initial letter of the word gravity. It is found both by theory and experiment that the value of g varies from place to place on the earth's surface. The value is least at the equator, and increases as we proceed from the equator to either pole. The difference between the value at the equator and the value at the pole is about $\frac{1}{200}$th of the value at the equator. Over such an area as that of the United Kingdom the variation is extremely small. At any point of that area the value may be considered to be 32·2 in feet and seconds, and 981 in centimetres and seconds.

In all cases of motion treated of hereafter, unless the contrary is expressly stated, the motion is supposed to take place *in vacuo*, so that the presence of the atmosphere need not be considered.

If m denote the mass of a body, and g the acceleration of gravity at a certain place, the weight of the body at that place in absolute units is by Article 37,

$$mg.$$

It follows that the weight of a given mass varies as the acceleration of gravity at the place where the observation is taken.

39. Gravitational Units of Force.

A gravitational unit of force is a unit defined as the weight of some standard mass, such as the mass of a pound or the mass of a gramme. We have seen in the preceding Article that a gravitational unit of force is not an invariable unit, as it varies with the acceleration of gravity. Thus the weight of a given body at the pole exceeds its weight at the equator by $\frac{1}{200}$th of its weight at the equator.

The student must carefully notice that the difference between the weights of a given mass at two points of the earth's surface could not be detected by an ordinary balance. In an ordinary balance a mass in one scale is balanced by an equal mass in the other scale, whatever be the acceleration of gravity. The difference, however, would be indicated by a spring-balance, in which the weight of the mass weighed pulls out a spring. It is evident that the greater the weight of the mass the farther will the spring be drawn out, that is, the greater will be the weight indicated by the balance.

It is often necessary in Dynamics to express the weight of a given mass at a certain place in terms of an absolute unit. To do this we require to know the acceleration of gravity at the place. By the preceding article the weight of a mass m is mg in absolute units. Hence the weight of a mass m is expressed in absolute units by multiplying by g, that is, we convert from gravitational units to absolute units by multiplying by the number representing g. Conversely, we convert from absolute units to gravitational units by dividing by the number representing g.

Absolute units of force are in use principally in Electricity and Astronomy. The weight of a pound, though not invariable, varies so little from point to point of the earth's surface, that it is adopted as a unit of force in many practical applications of Dynamics.

Example 1.—A spring-balance is graduated at London. What is the true mass of a body which weighs in this balance 115 pounds at Edinburgh?

Acceleration of gravity in C.G.S. units at London, 981·17; at Edinburgh, 981·54.

The weight of a given mass varies as the acceleration of gravity, and therefore,

$$\frac{\text{weight of the mass at London}}{\text{weight of the mass at Edinburgh}} = \frac{981{\cdot}17}{981{\cdot}54},$$

$$\text{or weight of mass at London} = \frac{115 \times 981{\cdot}17}{981{\cdot}54}$$

$$= 114{\cdot}96 \text{ nearly.}$$

Now since the balance is graduated at London, the weight of one pound at London is the unit weight, that is, the mass of one pound at London weighs one pound. Hence, since the weight of the body at London is 114·96, the mass is 114·96 pounds.

Example 2.—A certain force is equal to 953 poundals. Express this force in pounds weight at Glasgow, where $g = 32{\cdot}2$ in British units.

Since the acceleration of gravity is 32·2 at Glasgow, the weight of the mass of one pound at Glasgow is equal to 32·2 poundals. Hence the force of 953 poundals

$$= \frac{953}{32{\cdot}2}$$

$$= 29{\cdot}6 \text{ lbs. weight at Glasgow.}$$

40. Law III.

To every action there is an equal and opposite reaction.

The word action in this law means the force with which one body presses on, attracts, repels, or draws another body; and the law asserts that the second body presses on, attracts, repels, or draws the first body with an equal force. We give the following illustrations of this law:—

(1.) If a person exerts a downward force on a table with his hand, the table presses his hand upward with an equal force.

(2.) The sun attracts a planet with a certain force, and the planet attracts the sun with an equal force. On account of the mass of the sun being almost infinitely great compared with the mass of the planet, the effect of the planet's attractive force on the sun is inappreciable compared with that of the sun's attractive force on the planet.

(3.) A pair of pith balls, electrified with the same kind of electricity, repel each other with equal forces.

(4.) In a tug of war the tension of the rope between the two teams is the force acting on each team. Hence one team draws the other with the same force that this latter draws the former. The winning team overcomes the other by offering a greater resistance to being moved.

41. The laws of motion may be considered to be generalizations from observation and experiment. One of the strongest reasons, however, for believing the laws of motion to be true is indirect. The science of Physical Astronomy rests on Dynamics, and Dynamics rests on the three laws of motion. Now the places of the sun and the moon among the stars, calculated for years beforehand by the principles of Physical Astronomy, are verified every day in the observations made in observatories and by navigators at sea. It is inconceivable that innumerable complicated calculations should be found to agree with observation, if the laws on which these calculations ultimately depend were not strictly true.

EXAMPLES X.

[In the following set of Examples g is to be taken equal to 32·2 in feet and seconds, and equal to 981 in centimetres and seconds.]

1. What force—in pounds weight and in poundals—acting for 11 minutes will give a railway engine, whose mass is 15 tons, a velocity of $24\frac{3}{20}$ miles per hour?
2. A body whose mass is 20 lbs., moving with a velocity of 40 feet per second, is brought to rest by a constant resistance after passing over 400 feet. Find the magnitude of the resistance in poundals and in pounds weight.
3. A mass of 1 lb., acted on by gravity, is drawn upwards by a force of 33 poundals. What is the acceleration?
4. A cage of mass 1 ton is lowered down a pit with an acceleration of 16. Find the tension of the rope in poundals.

   A mass of 10 lbs. lies at the bottom of the cage. With what force does this mass press on the bottom of the cage?
5. A train is starting from a station with a constant acceleration of 5 feet per second per second. Compare the tension of the coupling between the engine and the first carriage with the weight of all the carriages.
6. A body is acted on for 3 seconds by a force equal to $\frac{1}{4}$ of its own weight. Show how far it will move while the force acts, and what velocity it will acquire.
7. If it were possible to diminish the mass of a body by one-half without altering its weight, show what would be the effect upon the rate at which its velocity would change when it was allowed to fall freely under the action of gravity.
8. A mass is moving under the action of a force with an acceleration of 135 centimetres per second per second. Find the force in dynes, and compare it with the weight of the mass.

CHAPTER IV.—GENERAL PRINCIPLES OF STATICS.

42. Range of Statics.

We have seen that the effect of a force acting on a body is to set the body in motion. Now there are many cases in which the motion of a body acted on by a force is prevented by the action of another or of other forces. In such cases, then, we have two or more forces acting on a body, which remains at rest. The consideration of such cases belongs to **Statics**, the science of the balancing or equilibrium of forces.

It is convenient to divide the problems which occur in Statics into two classes.

To the first class of problems belong those in which the directions of all the forces pass through a point in the body. In some problems of this class the body which is acted on by the forces is so small that its magnitude may be altogether neglected. Such a body would be like a mathematical point in having position but not magnitude, but it would differ from a mathematical point in having mass, and consequently in having the property of being acted on by force. Such a body is called a **material particle**, or simply a **particle**. To the same class of statical problems belong those in which the bodies are of appreciable magnitude, but the directions of the forces all pass through one point.

In the second class of problems in Statics we have to consider the action on bodies of appreciable magnitude of forces which do not all pass through the same point.

In problems in which the action of forces upon bodies of finite magnitude is considered, we shall always suppose that the bodies are perfectly **rigid**, that is, that they are changed neither in shape nor in size by the forces which act upon them. Now no body in nature is perfectly rigid, so that the science of Dynamics is founded on an abstraction, and hence our subject is sometimes called Theoretical or Abstract Dynamics. The solution of any problem on the principles of Abstract Dynamics will therefore be only a first approximation to the true solution, which can only be obtained by taking into account the effect of the forces in deforming the bodies on which they act.

The science which treats of the action of forces in changing the shape or size of bodies is called the science of **Elasticity**.

43. Specifications of a Force.

A force is completely specified when we are given—

(1.) Its magnitude.

(2.) Its direction.

(3.) Its point of application, that is, the point of the body at which it acts.

44. Forces may be Represented by Straight Lines.

Like a velocity and an acceleration, a force may be represented by straight lines. We can draw the straight line from a point taken to represent the point of application of the force; we can draw the straight line in the direction of the force; and we can make the length of the straight line to contain as many units of length—the unit of length being chosen arbitrarily—as the given force contains units of force. If we have more than one force, the straight line representing each force will be drawn proportional to the force, and we shall thus have a **force** diagram drawn to scale.

The direction of a force is not completely determined when we are given only the straight line in which the force acts; we must know also in which direction the force acts. Thus, if AB is a straight line, a distinction is drawn between the direction AB and the direction BA. If we say that a force acts in the direction AB, we mean that it acts from A towards B. If we say that it acts in the direction BA, we mean that it acts from B towards A. In diagrams arrow-heads are often used to indicate in which direction along a straight line a force acts.

45. Different Kinds of Force.

In problems in elementary Statics we meet with one or more of the following kinds of force:—

(1.) The weights of bodies, due to the action of gravity, acting vertically downwards.

(2.) The pressure which two bodies in contact exert the one on the other.

(3.) The tension or pull of a stretched rope.

46. Resultant, Equilibrant.

If a system of two or more forces acts on a body, and if a single force can be found which is equivalent in effect to the given system, then that force is called the **resultant** of the given system.

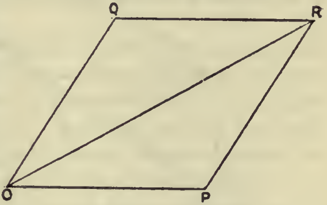
If a system of forces acts on a body, and if a single force can be found which, acting along with the given system, will keep the body at rest, then that single force is called the **equilibrant** of the given system of forces.

I.

FORCES IN ONE PLANE PASSING THROUGH A POINT.

47. Parallelogram of Forces.

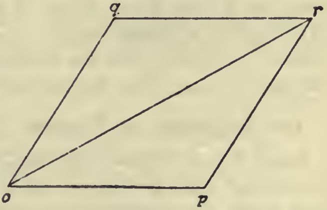
Like velocities and accelerations forces follow the parallelogram law. Thus we have, corresponding to the parallelogram of velocities and parallelogram of acceleration, the following proposition, known as the **Parallelogram of Forces**:—



If two forces, represented in magnitude and direction by the lines OP and OQ, act on a particle at O, and if the parallelogram OPRQ be completed, then the resultant of the two forces is represented in magnitude and direction by that diagonal OR which passes through O.

This proposition follows from the parallelogram of velocities with the help of Newton's second law of motion.

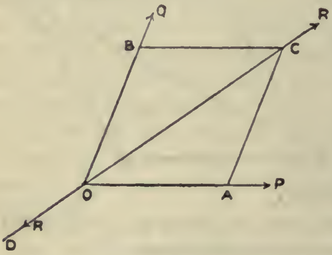
Let a *velocity* diagram *oprq* be drawn, *op* and *oq* representing the changes of velocity produced in unit of time by the forces represented in the *force* diagram by the lines OP and OQ. Then by the parallelogram of velocities the resultant change of velocity is represented by *or*. Hence *or* represents the resultant change of velocity in one unit of time produced by the action of the two forces represented by OP and OQ. But by Newton's second law of motion forces are parallel and proportional to the changes of velocity produced by them in unit of time. Hence the lines OP, OQ, and the line repre-



senting the resultant, are parallel and proportional respectively to *op*, *oq*, *or*. Hence the line which represents the resultant force in the *force* diagram is the line which corresponds to *or* in the *velocity* diagram; and that line is evidently OR, the diagonal of the parallelogram OPRQ.

48. Equilibrant of Two Forces acting on a Particle.

We have just seen that the resultant of the forces P and Q represented by OA and OB is a single force R represented by OC. We may therefore suppose P and Q to be replaced by R. Then the equilibrant of the force R can only be an equal force R acting at O in the opposite direction OD. For a force in any other direction than OD would, by the parallelogram of force, have a resultant with a force in the direction OC. Hence the equilibrant must act along OD, and evidently it must be equal to R. Hence we say that the equilibrant of the two forces P and Q is a force R equal in magnitude to the resultant of P and Q, and in a direction opposite to that of the resultant of P and Q. Hence, if the particle O were acted on by the three forces, P along OA, Q along OB, and R along OD, it would remain at rest. This is sometimes expressed by saying that the three forces, P along OA, Q along OB, and R along OD, are *in equilibrium*.



49. Triangle of Forces. Polygon of Forces.

Since forces follow the parallelogram law, all the consequences of that law given in Articles 19 to 24 will still be true when we substitute the word force for the word velocity. Thus we have propositions known as the **triangle of forces** and the **polygon of forces** corresponding respectively to the triangle of velocities and the polygon of velocities.

The triangle of forces is—

If three forces, represented in magnitude and direction by the sides of a triangle taken in order, act on a particle, then the forces are in equilibrium.

This may be proved independently thus:—

In the force diagram of the preceding Article the forces P along OA, Q along OB, and R along OD are in equilibrium. But these three forces are represented in magnitude and direction by OA, OB, OD (taken equal to OC), or, since AC is equal and parallel to OB, by OA, AC, CO. Hence three forces represented in magnitude and direction by the three sides OA, AC, CO of the triangle OAC would keep a particle at rest.

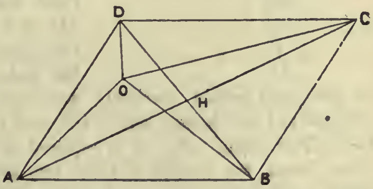
The polygon of forces is—

If any number of forces acting on a particle are represented in magnitude and direction by the sides of a closed polygon taken in order, then the forces are in equilibrium. If the lines representing the forces do not form a closed polygon, the resultant will be represented by the line joining the particle with the extremity of the last-drawn side of the incomplete polygon.

We have thus a complete graphical solution of the composition of forces passing through a point, and also a criterion of equilibrium.

The student will notice that we can find graphically the resultant of any number of forces acting in a straight line by means of the polygon of forces.

Example 1.—From any point O lines are drawn to the angular points of a parallelogram ABCD. If OA, OB, OC, OD represent graphically four forces acting at O, show that their resultant passes through the intersection of the diagonals of the parallelogram, and find its magnitude.



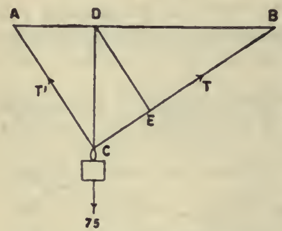
Taking the two forces represented in magnitude and direction by OA and OC, we find the resultant by completing the parallelogram of which OA and OC are adjacent sides, and drawing the diagonal which passes through O. Now this diagonal will pass through H, the middle point of AC the other diagonal, and will be equal in length to twice OH. Hence the resultant of the forces represented by OA and OC will be a force in the direction of OH, represented by a line equal in length to twice OH. In the same way we should find that the resultant of the forces represented by OB and OD is a force in the direction of OH represented in magnitude by twice OH. Hence the resultant of all the forces represented by OA, OB, OC, OD is a force in the direction of OH represented by a line equal in length to four times OH.

Example 2.—Show that three forces represented by the numbers 2, 4, and 8, cannot be in equilibrium.

Three forces can only be in equilibrium when they are proportional to the sides of some triangle. Now we cannot have a triangle whose sides are proportional to the numbers 2, 4, 8, since two of the sides, the sides proportional to 2 and 4, would be together less than the third, which is impossible (Euclid, I. 20).

Example 3.—A weight of 75 lbs. is hung by two strings, 3 feet and 4 feet long respectively, from two nails on the same level. If the distance between the nails is such that the strings are at right angles to each other, show what the tension in each string is.

Let A and B represent the positions of the nails, and C the position of the weight, which, for the purposes of this problem, may be considered as a particle. Let the vertical line through C meet AB in D; and draw DE perpendicular to BC. Then since AC, the shorter string, is 3 feet long, BC, the longer, 4 feet long, and since ACB is a right angle, it follows (Euclid, I. 47) that the hypotenuse AB is 5 feet long. The forces acting on the mass of 75 lbs. are the weight of the mass acting vertically downwards, the pull or tension T (suppose) of the longer string CB, and the tension T′ (suppose) in the shorter string CA. It is required to find T and T′. Since the three forces, the weight and the two tensions, are in equilibrium, they must be proportional to the sides of the triangles, whose sides are parallel to the directions of the forces.



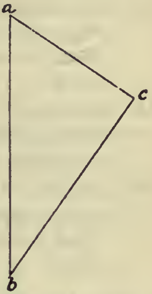
Drawing a triangle *abc* whose sides are parallel to the forces, we have a force diagram in which *ab* represents the weight, 75 lbs., *bc* the tension T in the string BC, *ca* the tension T′ in the string CA. Hence

$$\frac{T}{75}=\frac{bc}{ab}=\frac{CE}{CD}\left\{\begin{array}{l}\text{since triangles DCE and } abc\\ \quad\text{have their sides parallel,}\end{array}\right.$$

$$=\frac{AC}{AB}\left\{\begin{array}{l}\text{since triangles DCE and}\\ \quad\text{ABC are similar,}\end{array}\right.$$

$$=\frac{3}{5};$$

from which $T=45$ lbs.



Similarly it may be found that

$$T'=60 \text{ lbs.}$$

Hence the tension in the longer string is equal to the weight of 45 lbs., and the tension in the shorter string is equal to the weight of 60 lbs.

EXAMPLES XI.

1. ABDC is a parallelogram, and AB is bisected in E. Prove that the resultant of forces represented by AD, AC is double of the resultant of those represented by AE and AC.
2. Forces represented in magnitude and direction by the sides AB, BC, CD, and AD of a quadrilateral act on a particle. Find their resultant.
3. ABCD is a quadrilateral. Show that forces, acting on a particle, represented in magnitude and direction by AC, CB, BD, DA, would be in equilibrium.
4. Forces represented in magnitude and direction by the sides AB, AD, CB, CD of a parallelogram, act on a particle. Show that they are in equilibrium.
5. A system of forces acts on a particle. Show that if all the forces of the system are increased or diminished in the same ratio, the resultant is increased or diminished in the same ratio.
6. A system of forces acts on a particle, and is in equilibrium. Show that the system will still be in equilibrium if all the forces are increased or diminished in the same ratio.
7. Forces of 1, 2, 3, 4, 5, 6 units act from the centre O of a regular hexagon towards the angular points A, B, C, D, E, F respectively. Find the resultant.

8. How must forces of 7, 10, and 3 units be adjusted if they act at a point and are in equilibrium?
9. Two strings equally inclined to the vertical and making a right angle with each other, support a weight of 100 lbs. attached to them at their point of meeting. Find the tension of each string.
10. The ends of a cord 26 feet long are attached to two points on the same level 24 feet apart, and a weight of 50 lbs. is hung at the middle point of the cord. Find the tension of the cord.
11. A weight of 10 lbs. is suspended by a string, and is then displaced so that the string makes an angle of 60° with the vertical. What horizontal force will keep the weight in its new position?

50. Algebraical Formulæ.

Since forces follow the parallelogram law the algebraical formulæ given in Arts. 19 and 20 for velocities, hold good also for forces. Thus if P and Q represent two forces acting on a particle in directions inclined at an angle i, and if R represent their resultant, then

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos i}.$$

If θ represents the angle which the resultant makes with the force P, we have also

$$\tan \theta = \frac{Q \sin i}{P + Q \cos i}.$$

In the particular case in which the two forces are at right angles, $i = 90°$, and therefore $\cos i = 0$, $\sin i = 1$. Hence the resultant of two forces P and Q at right angles is given by the formula

$$R = \sqrt{P^2 + Q^2}.$$

Also the angle θ which this resultant makes with the direction of P is given by

$$\tan \theta = \frac{Q}{P}.$$

EXAMPLE 1.—If two forces represented by 12 and 5 act on a point at right angles to each other, find the resultant and the sines of the angles made by it with each of its components.

Here $\quad P=12,\ Q=5,\ i=90°;$

hence $\quad R=\sqrt{12^2+5^2}=\sqrt{169}=13.$

Also $\quad \tan\theta=\frac{Q}{P}=\frac{5}{12},$

from which $\quad \sin\theta=\frac{5}{13}$, and $\cos\theta=\frac{12}{13}.$

The magnitude of the resultant is 13, and the sines of the angles it makes with P and Q, or $\sin\theta$ and $\cos\theta$, are respectively $\frac{5}{13}$ and $\frac{12}{13}$.

Example 2.—If two forces of 27 lbs. and 63 lbs. act on a particle at an angle whose cosine is $-\frac{11}{14}$, find their resultant.

Here $\quad P=27,\ Q=63,\ \cos i=-\frac{11}{14};$

therefore
$$R^2=27^2+63^2+2\,.\,27\,.\,63(-\tfrac{11}{14})$$
$$=2025;$$

taking the square root, we find R the resultant = 45 lbs.

Example 3.—The resultant of two equal forces acting upon a point at an angle of 60° is 90 lbs. Find the magnitude of each component.

Let P denote the magnitude of each component; then, putting $P=Q$, $i=60°$, $R=90$, in the formula for the resultant, we get

$$P^2+P^2+2P^2\cos 60°=90^2,$$

from which $\quad 3P^2=90^2,$

or $\quad P^2=2700;$

taking the square root, $\quad P=30\sqrt{3}$ lbs.

EXAMPLES XII.

1. Two forces represented by the numbers 3 and 5 have a resultant 7. Find the angle between the forces.
2. Find the resultant of two forces of 60 and 91 lbs. respectively, which are at right angles to each other.
3. Two forces of 143 and 120 lbs. act at an angle of 120°. Find their resultant.
4. Two forces act at an angle of 60°. Their resultant is 151 lbs., and one of the forces is 56 lbs. Find the other force.
5. Three forces, 1·5, 2, 2·5 units respectively, act on a particle and are in equilibrium. Find the angle between the directions of the first two forces.

6. Prove that if two equal forces, each equal to P, act on a particle in directions inclined at an angle i, the magnitude of their resultant is $2P\cos\frac{i}{2}$.

7. Two equal forces applied at a given point have a resultant given in magnitude and direction. Find the locus of the extremity of the straight line which represents either force.

8. Two forces, P and 2 P, act on a particle in directions inclined at an angle of 120°. Find the magnitude and direction of the resultant.

9. Given two forces, show at what angle they must act in order that their resultant (1) may be least, (2) may be greatest.

10. Show geometrically or by means of the algebraical formula that the resultant of two given forces decreases when the angle between the forces is increased. What are its greatest and least values respectively?

11. If a heavy box be wholly suspended by a strap attached to the handles at its ends, show that the tension is diminished if the length of the strap is increased.

12. Two forces act at an angle of 60°, and the resultant divides the angle between them in the ratio of 3 to 1. If the smaller force is 8 poundals, find the larger, given $\sin 15° = \cdot 259$.

13. If two forces P and Q act upon a particle (1) when the angle between their directions is 60°, (2) when it is 120°, and if R and R_1 are the resultants in these two cases, prove that

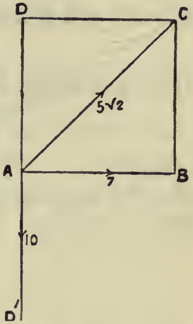
$$R^2 + R_1^2 = 2(P^2 + Q^2).$$

51. Algebraical Method of finding the Resultant of any number of Forces acting on a Particle.

Since forces follow the parallelogram law, it follows that the method of resolving a velocity into two components, explained in Articles 22 and 23, applies also to forces. Thus, if F be the measure of a force, and α the angle which the direction of the force makes with a given direction, the component in the given direction is $F\cos\alpha$, and the component in the direction at right angles to the given direction is $F\sin\alpha$.

Also the method of finding algebraically the magnitude and direction of any number of component velocities, explained in Article 24, will apply to forces.

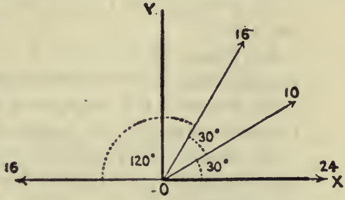
Example 1.—A, B, C, D are the angular points of a square taken in order. Find the resultant of the three forces, 7 from A to B, 10 from D to A, $5\sqrt{2}$ from A to C.



Take as the two directions at right angles the directions AB and AD'.

The force $5\sqrt{2}$ along AC has a component $5\sqrt{2}\cos 45^\circ$, or 5, along AB, and a component $5\sqrt{2}\cos 135^\circ$, or -5, along AD'. Hence the sum of all the components along AB is $7+5$ or 12, and the sum of all the components along AD' is $10-5$ or 5. Hence the three given forces are equivalent to two forces, one 12 along AB, and the other 5 along AD'. By the formula of Article 50 the resultant is found to be 13.

Example 2.—Four forces of 24, 10, 16, 16 units act upon a particle, the angle between the first and second being 30°, the angle between the second and third 30°, and the angle between the third and fourth 120°. Find the magnitude of the resultant and the tangent of the angle it makes with the first force.



Take OX, the direction of the first force, and OY, the direction at right angles to OX, as the two directions along which the forces are to be resolved.

The components of the force 24 along OX and OY are respectively

24 and 0.

The components of the second force 10 along OX and OY are respectively

$$10\cos 30^\circ \text{ and } 10\sin 30^\circ,$$

or

$$5\sqrt{3} \text{ and } 5.$$

The components of the third force 16 are respectively

$$16\cos 60^\circ \text{ and } 16\sin 60^\circ,$$

or

$$8 \text{ and } 8\sqrt{3}.$$

The components of the fourth force are respectively

$$-16 \text{ and } 0.$$

The sum of all the components in the direction OX is

$$24+5\sqrt{3}+8-16, \text{ or } 16+5\sqrt{3}.$$

The sum of all the components in the direction OY is

$$5+8\sqrt{3}.$$

Hence if R denote the magnitude of the resultant,

$$R^2=(16+5\sqrt{3})^2+(5+8\sqrt{3})^2$$
$$=548+240\sqrt{3};$$

therefore $R=\sqrt{548+240\sqrt{3}}$.

Also if θ denote the angle the direction of the resultant makes with OX,

$$\tan\theta=\frac{5+8\sqrt{3}}{16+5\sqrt{3}}.$$

EXAMPLES XIII.

1. Find in each of the following cases the magnitude of the resultant:—

(1.) Forces 5, 10, 15, 20, acting respectively east, north, west and south.

(2.) Forces of 1, 2, 3, 4, 5, 6, acting from the centre of a regular hexagon towards the angular points.

(3.) Forces 3, 4, 5, acting on a particle parallel respectively to the sides taken in order of an equilateral triangle.

(4.) Forces 3, 5, 7, acting on a particle at the centre of a circle towards points on the circumference which divide it into three equal parts.

(5.) Forces 30, 40, 40, acting on a particle, the angle between the directions of the first two being 45°, and that between the directions of the last two being 30°.

(6.) Forces P, P, 2P, acting on a particle, parallel to the sides of an equilateral triangle.

(7.) Forces of $3\sqrt{2}P$, 4P, 5P, 6P and 2P, acting on a particle, the directions of the last four forces making angles of 45°, 135°, 225°, and 315° respectively with the first force.

2. Find, with the help, when necessary, of a table of trigonometrical ratios, the angle which the resultant makes with the first force in each of the seven cases of the preceding Example.

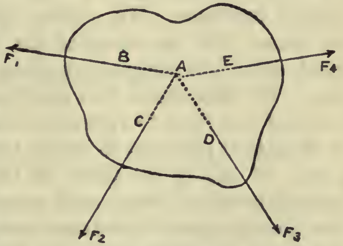
3. Two forces, P and Q, act on a particle; also a third force, P + Q, whose direction bisects the angle $2i$ between P and Q. Find the magnitude of the resultant.

4. Draw an isosceles triangle ABC, with angle A equal to 120°, and draw AD perpendicular to BC. A force P acts from A to B, a force 2P from A to C, and a force 2P from D to A. Find the magnitude of the resultant.

5. Four forces of 18, $12\sqrt{2}$, 50, $3\sqrt{2}$, and $12\sqrt{2}$ act respectively E., N.E., N., N.W., and W. Find the effective eastern component.

52. Rigid Body acted on by Forces whose lines meet.

When a rigid body is acted on by forces whose lines of action, produced if necessary, all meet in a point, we may suppose all the forces to act at that point, and the resultant will be found in the same way as if all the forces acted on a particle at the point.



Let ABCDE be a rigid body acted on by forces F_1, F_2, F_3, F_4, acting respectively at the points B, C, D, E, and let the lines of action of the forces, produced if necessary, meet at the point A. Then we may suppose that all the forces F_1, F_2, F_3, F_4 act at the point A, and the resultant will be a single force passing through the point A.

53. Conditions of Equilibrium. Application to Problems.

If X and Y denote the sums of the components of a given system of forces passing through a point resolved along two chosen directions at right angles, and if R denote the resultant, then by Article 50,

$$R^2 = X^2 + Y^2.$$

From this equation we can deduce the conditions that the given system of forces shall be in equilibrium. The system will be in equilibrium if the resultant is zero, that is, R must be equal to zero. Hence we must have

$$0 = X^2 + Y^2.$$

Hence the sum of the squares of X and Y must be zero. Since the square of any quantity is always positive, the sum of two squares cannot be zero unless each of the quantities that are squared is zero. Hence from the last equation we infer that

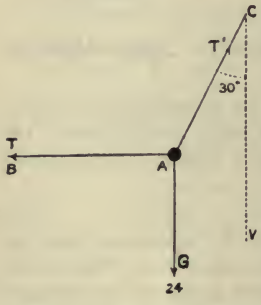
$$X = 0 \text{ and } Y = 0.$$

Now since the two directions along which each force was resolved were chosen arbitrarily, it follows that for equilibrium the following conditions must be satisfied:—

The sums of the components of all the forces along any two chosen directions at right angles must be zero.

These conditions give a simple method of solving many statical problems. We express in algebraical symbols the conditions that the sums of the components of the forces which enter into the problem along two chosen directions are zero. We thus obtain two algebraical equations, and it will be found that these two equations are sufficient for the solution of all statical problems in which the forces all pass through a point. If one of the forces in the problem is the force of gravity, it will be found most convenient to choose as the two directions along which the forces are to be resolved, the vertical line and the horizontal line in the plane in which the forces act.

EXAMPLE 1.—A ball weighing 24 lbs. is kept at rest by two strings, one of which is horizontal, and the other inclined to the vertical at an angle of 30°. Find the tensions of the two strings.



Let A be the ball kept at rest by the horizontal string AB and the string AC inclined to the vertical CV at an angle ACV of 30°.

For the purposes of this problem we may neglect the size of the ball, and consider it simply as a material particle at A. The ball is acted on by three forces—(1) its own weight, 24 lbs. in the line AG, that is, vertically downwards; (2) the tension T of the horizontal string AB; (3) the tension T′ of the string AC.

These three forces are in equilibrium, and therefore, by the conditions of this Article, the sums of all the horizontal components, and of all the vertical components, must be zero. The algebraical sum of the horizontal components is

$$T - T' \cos 60^\circ,$$

and the algebraical sum of the vertical components is

$$24 - T' \cos 30^\circ.$$

Hence T and T′ are to be found from the two equations,

$$T - T' \cos 60^\circ = 0.$$
$$24 - T' \cos 30^\circ = 0.$$

From the second equation we find

$$T' = \frac{48}{\sqrt{3}} \text{ lbs.},$$

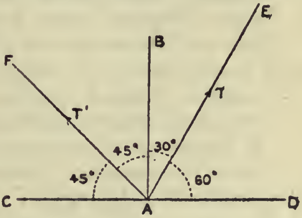
which, substituted in the first equation, gives

$$T = \frac{24}{\sqrt{3}} \text{ lbs.}$$

Hence the tensions in the two strings have been determined.

Example 2.—A boat is moored in a stream by two ropes, fixed to posts on the banks, and inclined to the direction of the current at angles of 30° and 45°. Find the ratio of the tensions of the two ropes.

For the purposes of this problem we may neglect the size of the boat, and consider it merely as a particle. Let A represent the position of the boat considered as a particle, and let T and T′ be the tensions of the two ropes AE and AF, these lines being inclined at angles of 30° and 45° respectively to the direction of the current BA. The weight of the boat, a force vertically downwards, is balanced by the upward pressure of the water, and may be left out of account in our problem. It is evident then that, as far as forces in the horizontal plane of the boat are concerned, the boat is in equilibrium under the three forces, the two tensions and the force exerted by the current on the boat. In order to find the ratio of the two tensions we resolve the forces along the line CAD, a line at right



angles to BA, the direction of the current, and equate the sum of the components to zero. The force of the current has no component along CD, since the direction of the current is at right angles to CD, and the algebraical sum of the components is

$$T \cos 60^\circ - T' \cos 45^\circ.$$

Hence $$T \cos 60^\circ - T' \cos 45^\circ = 0,$$

from which $$T - T'\sqrt{2} = 0,$$

or $$\frac{T}{T'} = \frac{\sqrt{2}}{1},$$

which gives the ratio of the two tensions.

EXAMPLES XIV.

1. A weight of 25 lbs. is suspended by two strings, inclined to the vertical at 30° and 60°. Find the tension of each string.

2. Four strings are knotted together. The ends of three of these, each of which is 13 feet long, are attached to three points, A,B,C, in the same plane, which form the angular points of an equilateral triangle. To the end of the fourth string a mass of 20 lbs. is hung. Find the tension in each of the upper strings, if the knot is 5 feet below the plane of the triangle ABC.

3. Five strings are knotted together. The ends of four of these, each of which is 101 inches long, are attached to four points, A, B, C, D, in the same plane, which form the angular points of a square. To the end of the fifth string a mass of 50 lbs. is hung. Find the tension in each of the upper strings, if the knot is 95 inches below the plane of the square ABCD.

4. A buoy is moored in a stream by two ropes inclined at a right angle, and fastened to points on the bank, the line joining the two points being perpendicular to the direction of the stream. If the force of the stream on the buoy is double the tension of one of the ropes, show that the length of the other rope is half the breadth of the stream.

5. The two ends of a string are fastened to two points A and B in the same horizontal line, the length of the string being twice the length of AB. A weight of 100 lbs. is fastened at a point P of the string. If P is so chosen that the angle APB is 60°, find by graphical construction the tensions of the two parts of the string.

II.

FORCES NOT PASSING THROUGH A POINT.

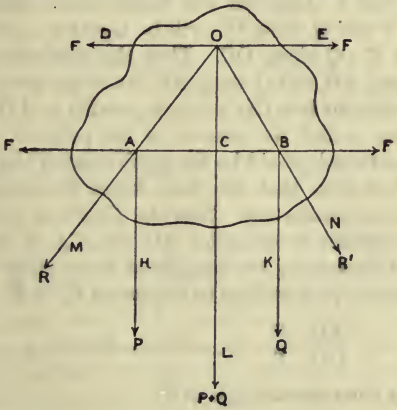
54. Parallel Forces.

The simplest case of forces not passing through the same point is the case in which a rigid body is acted on by parallel forces. If two or more parallel forces act in the same direction they are said to be **like** or **similar** parallel forces. If two parallel forces act in opposite directions they are said to be **unlike** or **dissimilar** parallel forces.

55. Resultant of two similar Parallel Forces.

Let P and Q be two similar parallel forces acting at the points A and B of the rigid body ACB in the directions AH and BK. Required to find their resultant.

Without altering the effect of the forces P and Q we may superimpose two forces each equal to F acting in the straight line ACB, one in the direction AB and the other in the direction BA. These two equal forces will balance each other, and hence the two given forces P and Q are together equivalent to the four forces P, Q, F along BA, and F along AB. Now the two forces P at A and F at A will by the parallelogram of forces have a resultant R in some direction



AM; and similarly the forces Q at B and F at B will have a resultant R′ in some direction BN. Hence the two given parallel forces are equivalent to the two forces R at A and R′ at B. Let the directions of these two forces R and R′ meet at O. Then we may suppose R and R′ to act at O. Thus the two given forces P and Q are equivalent to the forces R and R′ acting at O in the directions OA and OB. At O draw DOE parallel to ACB and OCL parallel to AH or BK. Resolve the force R into two components along the lines OD and OC, and the force R′ into two components along the lines OE and OC. It is evident that the component of R along OD and the component of R′ along OE will each be equal to F, and that the component of R along OC will be P and of R′ along OC will be Q. Hence the two given forces P and Q are together equivalent to the system of forces, F along OE, F along OD, P along OC, and Q along OC. The two forces F along OD and F along OE will balance each other, and the two forces P and Q along OC will be together equivalent to a single force P + Q along OC. Thus the resultant of the two forces P along AH and Q along BK is a single force P + Q acting in the straight line CL, which is parallel to AH and BK.

It remains to find the position of the point C. Since the force R, if reversed, would be the equilibrant of the two forces P and F, it follows that the three forces R reversed, P, and F would be in equilibrium. Now the directions of these three forces are parallel to the sides AO, OC, CA of the triangle AOC, and therefore by the triangle of forces these three sides are respectively proportional to the forces R, P, F.

Hence
$$\frac{AC}{CO} = \frac{F}{P} \quad \text{.........(1).}$$

Similarly from the triangle OCB
$$\frac{CO}{CB} = \frac{Q}{F} \quad \text{.........(2).}$$

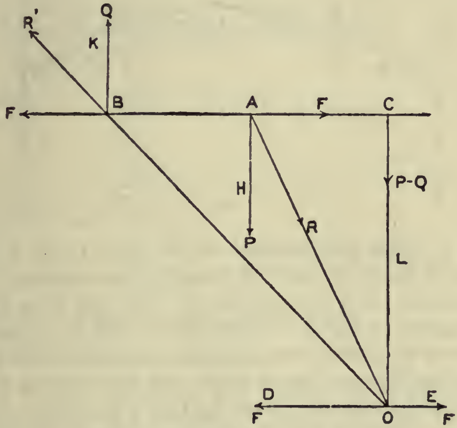
Hence
$$\frac{AC}{CB} = \frac{AC}{CO} \times \frac{CO}{CB} = \frac{F}{P} \times \frac{Q}{F} = \frac{Q}{P},$$

which determines the ratio of the two parts into which the line AB is divided at the point C. We express this result in words by saying that the point C divides the line AB into two segments AC and CB, which are inversely proportional to the forces adjacent to them respectively.

56. Resultant of two dissimilar Parallel Forces.

Let P and Q be two dissimilar parallel forces acting at points A and B of a rigid body. Required to find their resultant.

Let P be the greater force; then by the construction shown in the accompanying diagram, which is similar to that in the preceding Article, the student will have no difficulty in proving



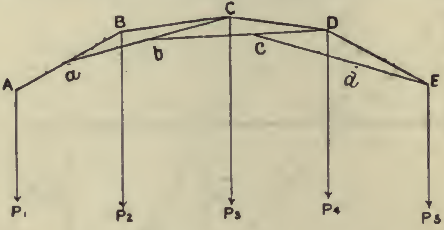
that the resultant is a single force $P - Q$, that it acts at a point C which lies on the line BA produced through the point A at which the greater force acts, and that this point C divides the line AB externally into two segments AC and BC which are inversely proportional to the forces adjacent to these segments.

57. Resultant of a System of Parallel Forces. Centre of Parallel Forces.

If a system of more than two parallel forces acts on a rigid body, the resultant is found by repeated application of the propositions of the two preceding Articles.

Let, for example, a system of five parallel forces, P_1, P_2, P_3, P_4, P_5, act at the points A, B, C, D, E respectively. To find the resultant we proceed thus:—

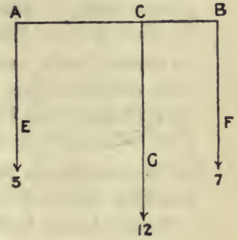
We first find the resultant of two of forces P_1 and P_2. This resultant will be a single force acting at some point a in the straight line AB. The forces P_1 and P_2 at A and B may therefore be replaced by their resultant acting at a, and this resultant combined with the force P_3 at C will give as resultant a single force acting at some point b between a and C. This force at b, which is equivalent to the three forces P_1, P_2, P_3, combined with the force P_4 at D will give as resultant a single force at some point c in the line bD; and this force at c will be equivalent to the four forces P_1, P_2, P_3, P_4. Lastly, the resultant of the force at c and the force P_5 at E will be a single force acting at some point d in cE, and this force at d will be the resultant of the five forces P_1, P_2, P_3, P_4, P_5.



It is to be noticed that the position of the point d does not depend in any way on the actual direction of the parallel forces: the position of d will remain unchanged though the directions of the forces are changed, if only the forces remain parallel. On account of this property d is called the *centre* of the five parallel forces. In general, whatever be the number of forces, the following proposition is true:—

If a system of parallel forces acts on a rigid body, each force passing through a given point in the body, then the resultant will always pass through a fixed point whatever be the direction of the parallel forces, and this point is called the centre of the given system of parallel forces.

Example.—Find the resultant in magnitude and direction of two forces of 5 and 7 units acting in parallel lines 2 feet apart, (1) when the forces are similar, (2) when they are dissimilar.



(1) Let the two forces act in the same directions, AE and BF, the distance AB between their lines of action being 2 feet.

The resultant will be a force acting in a line CG parallel to AE and BF. The magnitude of the resultant will be 5 + 7, or 12. The point C divides AB into two segments,

such that $\frac{AC}{CB}=\frac{7}{5}$, or $\frac{AC}{AC+CB}=\frac{7}{7+5}$,

or $\frac{AC}{AB}=\frac{7}{12}$.

But AB = 2 feet = 24 inches.

Hence $\frac{AC}{24}=\frac{7}{12}$, from which AC = 14 inches.

Hence the remaining segment BC is 10 inches.

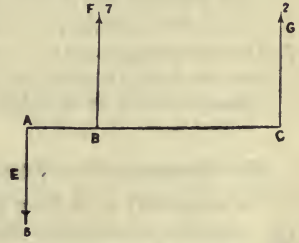
(2) When the forces are dissimilar.

The resultant in this case is a force in the direction CG, the direction of the greater force, and its magnitude is the difference of the two given parallel forces, that is, is 2 units. To find the point C, where it acts, we have the equation

$$\frac{AC}{BC}=\frac{7}{5},$$

from which $\frac{AC}{AC-BC}=\frac{7}{7-5}$,

or $\frac{AC}{AB}=\frac{7}{2}$,



or $\frac{AC}{24}=\frac{7}{2}$ or AC = 84 inches, and therefore BC = 60 inches.

EXAMPLES XV.

1. Find the resultant of two forces of 7 and 13 units acting in parallel lines 40 inches apart, (1) when the forces are similar, (2) when they are dissimilar.

2. Two parallel forces, each of 8 poundals, act in opposite directions, their distances apart being 15 inches, and in a line midway between their lines of action a third force of 16 poundals acts. Find the magnitude and line of action of the resultant.

3. Find the magnitude and line of action of the resultant of the following system of forces in the same plane:—

   Forces A and C, 7 and 8 units respectively, act vertically downwards, and forces B and D, 6 and 10 units respectively, act vertically upwards, the distance between A and B being 12 inches, the distance between B and C 4 inches, and that between C and D 9 inches.

4. ABC is a straight line; AB = 12 inches, BC = 18 inches. A force of 20 lbs. acts at C. Resolve it into two unlike parallel forces at A and B.

5. A system of 5 like parallel forces of 1, 2, 3, 4, 5 units acts at points in a straight line, the distance between two adjacent points being 6 inches. Find the resultant and the centre of the system of parallel forces.

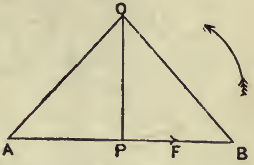
6. Find the resultant and centre of a system of 7 parallel forces of 1, 2, 3, 4, 5, 6, 7 units acting at points A, B, C, D, E, F, G respectively in a straight line. The forces at A, C, E, G act vertically downwards, and those at B, D, F vertically upwards. The distances AB, AC, AD, AE, AF, AG are respectively 8, 12, 17, 23, 30, 38 inches.

7. ABC is a triangle, and O is any point within it. Parallel forces act at A, B, C, which are proportional to the areas of the triangles OBC, OCA, OAB respectively. Show that the resultant passes through O.

58. Moment of a Force about a point.

The **moment** of a force about a point is the product of the numbers representing the magnitude of the force and the length of the perpendicular from the point on the line of action of the force.

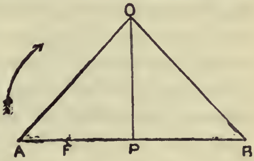
Thus if a force F is represented graphically by the straight line AB, the moment of the force F about the point O will be the product of the number of units of length in AB and the number of units of length in the perpendicular OP. Since the area of the triangle AOB is half the product of the perpendicular OP and the base AB, it follows that the moment of the force F about O is measured by twice the area of the triangle AOB.



It is evident, from the definition, that a force has no moment round a point in its line of action.

The tendency of a force to produce rotation round a point is measured by the moment of the force about the point.

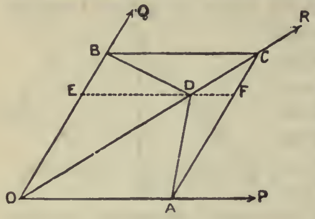
The student will notice that in the first figure the force F acts from A to B, and would, if O were a fixed point, tend to produce rotation round O in a direction contrary to that of the hands of a watch. In the second figure the force F acts from B to A, and would, if O were a fixed point, tend to produce rotation round O in the direction of the hands of a watch. In order to distinguish between the cases it is conventional to make use of the algebraical terms positive and negative, with their corresponding signs + and −. The moment round a point of a force tending to produce rotation in a direction contrary to that of the hands of a watch is usually called a positive moment, and consequently the moment of a force tending to produce rotation in the opposite direction a negative moment.



59. The Algebraical Sum of the Moments of two Forces about any point in the Line of Action of their Resultant is zero.

(1) Take first the case in which the lines of action of the two forces meet. Let OA and OB represent graphically the

two forces P and Q; then the resultant R is represented graphically by OC. If any point D is taken in the line OC, the moments of the two forces P and Q round O are measured by twice the areas of the triangles OAD and OBD, the area OAD being considered a positive quantity, and the area OBD a negative quantity. We prove the proposition by proving that these two areas are equal.



Draw EDF parallel to OA. Then, by Euclid I. 41,

$$\begin{aligned}\text{Area ODA} + \text{area BDC} &= \tfrac{1}{2}\,\text{area OAFE} + \tfrac{1}{2}\,\text{area EFCB}\\ &= \tfrac{1}{2}\,\text{area OACB},\end{aligned}$$

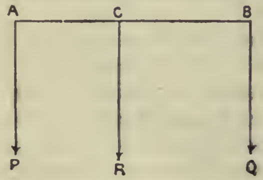
and $\quad \text{area OBD} + \text{area BDC} = \tfrac{1}{2}\,\text{area OACB};$

hence $\quad \text{area ODA} + \text{area BDC} = \text{area OBD} + \text{area BDC},$

from which $\quad \text{area ODA} = \text{area OBD}.$

Since then the two areas are equal and of opposite algebraical sign, their algebraical sum is zero.

(2) Take next the case of two parallel forces. Let P and Q be the two parallel forces, AB being the perpendicular distance between their lines of action. If C be the point where the line of action of the resultant R cuts AB, we have (Art. 55)



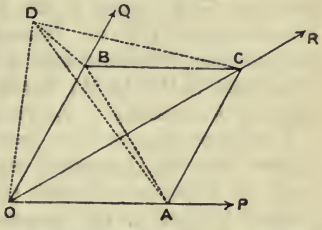
$$\frac{P}{Q} = \frac{BC}{AC};$$

from which $P \times AC = Q \times BC$.

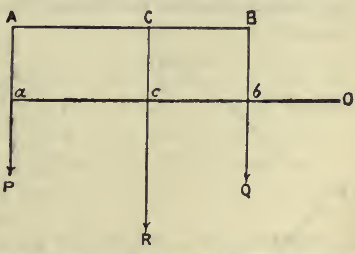
Now $P \times AC$ represents the moment of P round C, and $Q \times BC$ represents the moment of Q round the same point. Hence the two moments are equal, and it is evident that they are of opposite algebraical sign. Hence the algebraical sum of the two moments round C is zero. By the construction C is any point in the line of action of the resultant, and therefore the proposition is proved.

60. The Algebraical Sum of the Moments of two Forces round any point is equal to the Moment of their Resultant round the same point.

(1) Let the two forces P and Q meet in a point, and let them be represented graphically by OA and OB, the resultant R being represented graphically by OC. Let D be any point; then the moments of P, Q, and R are represented by twice the areas of the triangles OAD, ODB, and OCD. In the position in which D happens to lie in the figure all these moments are positive, and the proposition will be proved by proving that the sum of the areas OAD and OBD is equal to the area OCD. Since the altitude of the triangle ODA is equal to the sum of the altitudes of the triangles OBA and BDC, and since the bases are equal, it follows that the area of the triangle ODA is equal to the sum of the areas of OBA and DBC, that is, is equal to the triangle DBC and half the parallelogram. Hence the areas OAD and OBD are together equal to the sum of the areas DBC, OBD, and half the parallelogram, that is, to the three areas DBC, OBD, OBC, which make up the area OCD. Hence the areas ODA and OBD are together equal to the area OCD.



(2) Let the two forces P and Q be parallel, AB being their perpendicular distance apart, and let C be the point where the resultant R cuts AB. Let O be any point, and let *Obca* be drawn cutting the lines of action of the forces in the points *b*, *c*, *a*. Then in the position in which O lies in the figure all the moments round O are positive, and the pro-



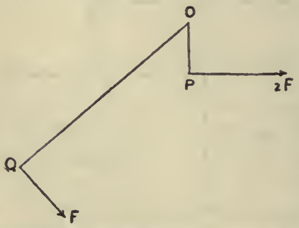
position will be proved by showing that the sum of the moments of P and Q round O is equal to the moment of R round O. The sum of the moments of P and Q is

$$\begin{aligned} & Q \times Ob + P \times Oa \\ & \quad = Q\,(Oc - bc) + P\,(Oc + ca) \\ & \quad = (P + Q)\,Oc + P \times ca - Q \times bc. \end{aligned}$$

Now, since c is a point in the line of action of the resultant, $P \times ca$ is equal to $Q \times bc$. Hence $P \times ca - Q \times bc = 0$, and therefore the sum of the moments of the two forces P and Q round O is equal to $(P + Q)\,Oc$, that is, is equal to $R \times Oc$, that is, is equal to the moment of the resultant round O.

The proposition is thus proved for two forces. It easily follows that the algebraical sum of the moments round any point of a system of forces passing through a point or all parallel is equal to the moment of the resultant round the same point. For the algebraical sum of the moments of any two is equal to the moment of the resultant of these two, and the algebraical sum of the moments of this resultant and a third will be equal to the moment of the resultant of all three, and so on.

This proposition affords a ready method of finding the line of action of the resultant of a system of parallel forces. First find the magnitude of the resultant by Art. 57. Next choose some point, and express that the moment of the resultant about that point is equal to the algebraical sum of the moments of all the forces about that point. This will give an algebraical equation for finding the position of the resultant.



EXAMPLE 1. — One force is double another, but the moment of the latter round a certain point is double that of the former. Compare the distances of the point from the lines of action of the two forces.

Let 2F and F be the two forces, and let OP and OQ be the perpendiculars on their lines of action from the given point O. Then, since the moment of the force F round O is double the moment of the force 2F round O, we have

$$F \times OQ = 2 \times 2F \times OP,$$

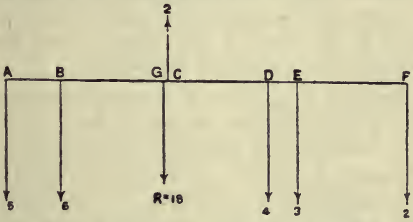
from which, by dividing both sides by F, we get

$$OQ = 4OP,$$

or, the distance of the line of action of the smaller force from the given point is four times the distance of the line of action of the larger force from the given point.

EXAMPLE 2.—Find the resultant of the following system of parallel forces:—

At points A, B, D, E, and F in a straight line AF, parallel forces of 5, 6, 4, 3, and 2 oz. respectively act in one direction, and at C another parallel force of 2 oz. acts in the opposite direction. AB = 6 inches, BC = CD = 14 inches, DE = 3 inches, EF = 16 inches.



It is evident that the magnitude of the resultant is

$$5 + 6 + 4 + 3 + 2 - 2 = 18 \text{ oz.}$$

The point, G suppose, where the line of action of the resultant cuts AF will be found by expressing algebraically that the algebraical sum of the moments of all the forces round any point is equal to the moment of the resultant round the same point. Take moments round the point A. The moment of the force 5 oz. at A round A is zero, the moment of the force 6 at B is $6 \times AB$ or 6×6, the moment of the force 2 at C is $-2 \times AC$ or -2×20, the moment of the force 4 at D is $4 \times AD$ or 4×34, the moment of the force 3 at E is $3 \times AE$ or 3×37, the moment of the force 2 at F is $2 \times AF$ or 2×53, and the moment of the resultant round A is $18 \times AG$; hence

$$\begin{aligned} 18 \times AG &= 6 \times 6 - 2 \times 20 + 4 \times 34 + 3 \times 37 + 2 \times 53 \\ &= 349 \end{aligned}$$

from which $AG = 19\frac{7}{18}$.

Hence the resultant is 18 oz. and acts in a line which cuts AF at a point G whose distance from A is $19\frac{7}{18}$ inches.

EXAMPLES XVI.

1. ABC is a triangle, AD being the perpendicular on BC. Show that the forces represented in magnitude, direction, and position by BA, BC, DA, DC exert no moment about the middle points of AC and BD.

2. A, B, C, D are the angular points of a square taken in order, and equal forces, each equal to P, act from A to B, from B to C, from C to D, and from D to A. If a represents the side of the square, show that the moment of the forces round any point in the plane of the square is $2Pa$.

3. Show generally that if forces act along the sides of any polygon taken in order, each force being represented by the side along which it acts, then the moment of the system of forces round any point in the plane of the polygon is the same, and is measured by twice the area of the polygon.

4. Parallel forces of 6, 5, 4 act in the same direction at points 4 inches, 6 inches, 9 inches respectively from one end of a line. Find the magnitude and position of the resultant.

5. Parallel forces of 2, 4, 6 units act in the same direction at points A, C, E respectively in a straight line ABCDE, and parallel forces of 3 and 5 units act in the opposite direction at points B and D of the same straight line. AB = 6 inches, BC = 7 inches, CD = 8 inches, DE = 9 inches. Find the magnitude, direction, and position of the resultant.

6. Find the resultant and the centre of parallel forces in each of the following cases:—

   (1) Parallel forces of 1, 2, 3, 2 units acting in the same direction at the angular points A, B, C, D respectively of a square, whose side is a.

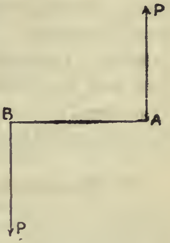
   (2) Parallel forces of 1, 5, 3, 5, 1, 3 units acting in the same direction at the angular points A, B, C, D, E, F respectively of a regular hexagon, whose side is a.

   (3) Four parallel forces, two of 1 unit and 5 units acting in one direction at the ends A and C of a diagonal AC of a square, two of 2 units and 2 units acting in the opposite direction at the ends B and D of the other diagonal, side of square being a.

61. Couples.

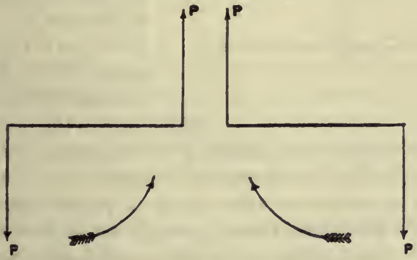
A **couple** in dynamics is a system of two equal parallel dissimilar forces.

If P represents either force of a couple, and AB the perpendicular distance between the lines of action of the two equal forces, then AB is called the **arm** of the couple, and the product of the force P and the arm AB is called the **moment** of the couple.



If we attempt to find by the method of Article 56 the resultant of the two forces forming a couple, we find that the magnitude of the resultant is zero, and that the construction for the point where it acts fails. It follows that a single force cannot be found which is equivalent to the two forces forming a couple. Now a single force acting on a body tends to cause a motion of translation in the straight line in which the force acts. The kind of motion which a couple causes or tends to cause must therefore be different from a motion of translation; it can therefore only be a motion of rotation.

An example of a couple is seen in the effect of the earth's magnetism on the two ends of a compass needle. The needle takes up a position of equilibrium in which it lies due magnetic north and south. If displaced from this position, the earth's magnetism attracts one end of the needle and repels the other with equal parallel forces, and these two forces form a couple tending to make the needle rotate back to its position of equilibrium.

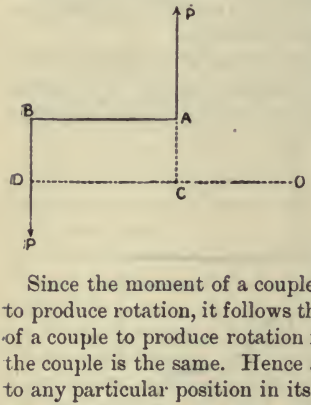


A distinction has to be drawn between couples which tend to cause rotation in a direction contrary to that of the hands of a watch and those which tend to cause rotation in the opposite direction. In the left-hand diagram a couple of the first kind is shown, and a couple of the opposite kind in the right-hand diagram. The difference between the two kinds of couples is indicated by the use of the algebraical terms positive and negative, with their corresponding signs + and −. It is usual to consider couples of the first kind as positive, and consequently those of the other kind as negative.

62. Elementary Propositions regarding Couples.

PROP. I.—*The algebraical sum of the moments of the two forces forming a couple round any point in the plane of the couple is equal to the moment of the couple.*

Let P, P be the two forces of the couple acting at the arm AB; let O be any point in the plane of the couple, and let OCD be drawn perpendicular to the direction of the forces. Then the algebraical sum of the moments of the forces round O is



$$
\begin{aligned}
&P \times OD - P \times OC \\
&= P(OD - OC) \\
&= P \times CD = P \times AB, \\
&\quad \text{(since } CD = AB\text{).} \\
&= \text{moment of the couple,}
\end{aligned}
$$

which proves the proposition.

Since the moment of a couple is the measure of its tendency to produce rotation, it follows that the measure of the tendency of a couple to produce rotation round all points in the plane of the couple is the same. Hence a couple has no special relation to any particular position in its plane, and it may therefore be moved about in its own plane from one position to any other without in any way altering its effect.

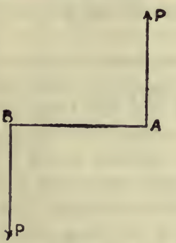
In moving a couple from one position to another in its own plane, we must be careful not to change the direction of rotation which would be caused by the couple.

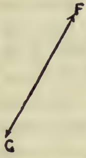
Moreover, since the only important characteristics of a couple are its moment and the direction of rotation it tends to cause, we can change the arm of the couple if at the same time we change the force so that the moment remains unaltered. Hence we are led to this result, that two couples are equivalent if the directions of rotation are the same, and if their moments are equal.

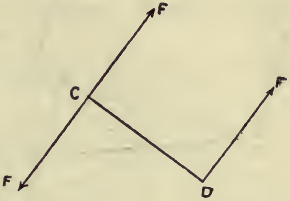
PROP. II.—*A single force and a couple are equivalent to a single force.*

Let the couple P, P and the force F act on a rigid body; the resultant shall be a single force F acting at some other point than C.

Without changing the moment of the couple P, P, change the force—and consequently the arm—to F, and rotate this couple in its own plane until one of the forces F is opposite to the force F. Instead then of the couple P, P acting at the arm







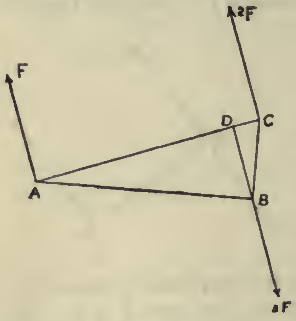
AB and the force F at C, we have the equivalent system of the couple F, F acting at the arm CD and the force F at C. The two forces F, F at C, acting in opposite directions balance each other, and we are left with the single force F acting at the point D. Hence the given force and the couple are equivalent to a single force.

From this proposition we deduce that a single force cannot balance a couple. For a single force and a couple always have a single force as their resultant, and they cannot therefore be in equilibrium.

PROP. III.—*The resultant of any number of couples acting in the same plane is a couple whose moment is equal to the algebraical sum of the moments of all the couples.*

Since the only characteristics of couples are their moments and the direction of rotation, it is evident that the resultant of two couples is a couple whose moment is equal to the sum of their moments if their directions of rotation are the same. and equal to the difference of their moments if their directions of rotation are opposite. Adopting the use of the algebraical signs + and −, as explained in Article 58, we express this by saying that the resultant of two couples is a couple whose moment is equal to the algebraical sum of the moments of the two couples. Similarly if we have more than two couples.

Example 1.—ABC is a triangle right-angled at B, AB being 2 feet long and BC 10 inches. At A a force F is applied at right angles to AC, at C a force 2F is applied in the same direction, and at B a force 3F in the opposite direction. Find the moment of the resultant couple.



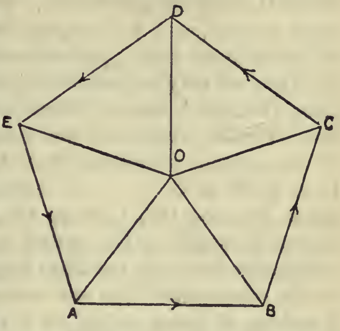
That the resultant is a couple is seen by considering that the force F at A and the force 2F at C have a resultant 3F acting at some point between A and C. This force, and the force 3F at B, constitute a couple. Let the direction of the force 3F at B cut AC at D, then BD is the perpendicular from the right angle B upon the hypotenuse. The moment of the couple is equal to the algebraical sum of the moments of the three forces round any point in the plane. Take the point A; the moment of F at A is zero, the moment of 3F at B is 3F × AD, and the moment of 2F at C is 2F × AC. Since the sides AB and BC of the right-angled triangle ABC are 24 inches and 10 inches respectively, it follows from Euclid that AC is 26 inches, and that AD is $\frac{288}{13}$ inches; here the moment of the resultant couple is

$$3F \times AD - 2F \times AC$$

$$= F\left(3 \times \frac{288}{13} - 2 \times 26\right) = \frac{188}{13}F.$$

Example 2.—Show that forces acting along and represented by the sides of a polygon taken in order are equivalent to a couple.

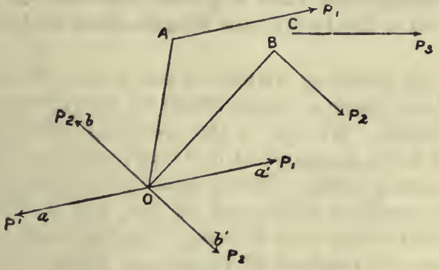
Let ABCDE be a polygon, the sides of which represent forces; it is required to show that this system of forces is equivalent to a couple. Take any point O and join O to the angular points of the polygon. The moment of the force represented by AB is twice the area of the triangle AOB, of the force BC by twice the area of the triangle BOC, and so on. Hence the sum of the moments of all the forces is twice the area of the polygon, and this result is true for any point in the plane of the polygon. Hence the system of forces must be equivalent to a couple.



63. Resultant of any System of Forces acting on a body in one plane.

Let P_1, P_2, P_3 .... be the system of forces acting on a body in the same plane at points A, B, C, ... respectively.

Take any point O in the plane, and superimpose two forces each equal to P_1, acting in opposite directions, Oa and Oa', the line aa' being parallel to the direction of P_1 at A. These two forces, being equal and opposite, would balance each other, and thus the effect of the system of forces would not be altered by the addition of these two forces. Now the three forces, P_1 at A, P_1 along Oa, and P_1 along Oa', are equivalent



to a single force P_1 along Oa' and a couple constituted by the forces P_1 along Oa and P_1 at A. We may therefore replace the single force P_1 at A by a single force P_1 along Oa' and a couple whose moment is P_1 × perpendicular from A on aa'. In the same way we may replace the single force P_2 at B by a force P_2 along Ob', and a couple, whose moment is P_2 × perpendicular from B on bb', and so on; and similarly with the other forces of the system. It follows that the system of forces P_1 at A, P_2 at B, P_3 at C . . . . is equivalent to a system of single forces, P_1 along Oa', P_2 along Ob', P_3 along Oc', ... and a system of couples. The system of single forces at A can be compounded into one resultant force, and the system of couples can be compounded into one resultant couple. It follows that a system of forces acting on a rigid body in one plane is equivalent to a couple in that plane, and a single force acting at any arbitrarily chosen point in the plane. The single force, being the resultant of forces P_1, P_2, P_3 . . . . , will always be the same whatever point is chosen. The moment of the resultant couple will evidently be equal to the algebraical sum of the moments round O of the forces of the system, and consequently the moment of the resultant couple will depend on the position of the point O.

64. Conditions that a given System of Forces acting on a Body in one Plane shall be in Equilibrium.

From the preceding Article we can deduce the conditions that a given system of forces acting on a rigid body may be in equilibrium. We saw in the preceding Article that the system of forces is equivalent to a single resultant force and a single resultant couple. Since a force cannot balance a couple, the single resultant force must be zero, and the moment of the resultant couple must be zero. Now the resultant force is the same as the resultant of a system of forces equivalent to the given system acting on a particle. Hence by Article 53 one condition is—

(1) *The algebraical sum of the components of all the forces along any two chosen directions at right angles must be zero.* Also we saw that the moment of the resultant couple is equal

to the algebraical sum of the moments of all the forces round O. For equilibrium this moment must be zero for all positions of O. Hence the other condition is—

(2) *The algebraical sum of the moments of the forces of the system round all points in the plane of the forces must be zero.*

The application of these conditions of equilibrium will give the solution of any problem which the student will meet with in elementary statics.

65. Constrained Body.

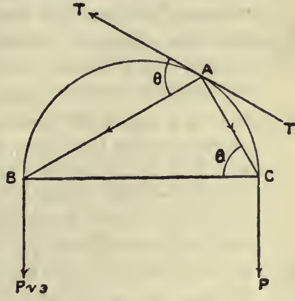
When the motion of a body is constrained, so that it can only move with a motion of translation or a motion of rotation, but not with both, then the conditions of equilibrium can be simplified.

If the body can move with a motion of translation in one direction only, the condition of equilibrium is that the sum of the component of the forces in the direction in which it is free to move shall be zero.

If a point in a body is fixed, so that the body can only move with a motion of rotation round that point, the condition of equilibrium is that the algebraical sum of the moments of the forces round that point shall be zero.

Example.—A smooth circular wire is fixed horizontally, and a small ring A can slide on it. A is attached to two strings, which pass through holes B and C at the extremities of a diameter, and support weights which are in the ratio of $\sqrt{3} : 1$. Prove that A will be at rest when CA makes an angle of 60° with BC.

The only motion the ring A can have is a motion along the curve CAB. Hence the condition of equilibrium is that the components of the forces along the tangent to the curve must be zero. Let the point A in the figure represent the position of equilibrium, and let θ denote the angle AC makes with BC. Since the weights at B and C are in the ratio of $\sqrt{3} : 1$, we may denote the weights by $P\sqrt{3}$ and P



respectively. These forces will cause tensions $P\sqrt{3}$ and P respectively in the two strings AB and AC, and these are the only two forces tending to produce motion along the curve. Hence for equilibrium the algebraical sum of the components of these two forces along the tangent TAT′ must be zero. Hence

$$P\sqrt{3}\cos BAT - P\cos CAT' = 0.$$

Now the angle BAT is equal to the angle ACB or θ, and the angle CAT′, the complement of BAT, is $90° - \theta$. Hence

$$P\sqrt{3}\cos\theta - P\sin\theta = 0$$

or, dividing by $P\cos\theta$,

$$\sqrt{3} - \tan\theta = 0, \text{ or } \tan\theta = \sqrt{3}.$$

Hence θ must be an angle of 60°, or AC makes an angle of 60° with BC.

EXAMPLES XVII.

1. Forces of 30 and 20 units act on a body in parallel but opposite directions, their lines of action being 6 feet apart. Midway between them a force of 10 units acts parallel to and in the same direction as the force of 20 units. Find the moment of the resultant couple.

2. ABCDE is a straight line, and at the points A, C, E similar parallel forces of 3, 5, 7 units respectively act perpendicular to AE, while forces of 13 and 2 units respectively act at B and D parallel to the former forces, but in the opposite direction. If AB, AC, AD, AE are respectively 3, 5, 7, 9 feet, find the moment of the resultant couple.

3. ABCD is a rectangle. Forces of 5, 3, 5, 3 units act from A to B, from C to B, from C to D, from A to D respectively. Show that these forces form a couple.

4. ABCD is a square whose side is $2a$. Forces of 16P, 4P, 4P, 9P act respectively from A to B, from B to C, from C to D, and from D to A. Show that this system of forces is equivalent to a single force 13P acting at the centre of the square and a certain couple. Find the moment of the couple and the tangent of the angle the direction of the single force makes with AB.

5. ABCD is a rectangle, AB being 4 feet, and BC 6 feet. A system of forces acts as follows:—A force of 6 units from A to B, a force of 16 units from C to B, a force of 12 units from C to D, and a force of 15 units from D to A. Reduce this system of

forces to a single force acting at the centre of the rectangle and a couple, finding the magnitude of the force and the moment of the couple.

6. ABC is a triangle, the sides BC, CA, and AB being 13 feet, 12 feet and 5 feet respectively. Forces of 12, 5 and 13 units act from A to B, from B to C, and from C to A respectively. Reduce this system to a couple and a single force acting at the centre of the circle inscribed in the triangle, finding the magnitude of the force and the moment of the resultant couple.

CHAPTER V.—THE FORCE OF GRAVITY.

SECTION I.—STATICS.

66. Centre of Gravity of a Body.

We saw (Article 38) that the force which causes a body to fall to the surface of the earth is a force of attraction, called the force of gravity, which the earth exerts on the body. We also saw that at the same place the amount of this force varies as the mass of the body. If then we conceive the body to be broken up into an infinite number of infinitesimal portions, each of these portions will be attracted to the earth with a force proportional to its mass, and we may consider the action of gravity on the body, that is, the weight of the body, as the resultant of an infinite number of forces. If the resultant of that system of forces always passes through a fixed point in the body, whatever be the position of the body with reference to the earth, then that fixed point is called the **Centre of Gravity** of the body.

67. Centre of Inertia.

If, as in the preceding Article, we conceive a body to be made up of an infinite number of infinitesimal portions, and if these portions are acted on by a system of parallel forces, each force being proportional to the mass on which it acts, then the centre of that system of parallel forces is called the **Centre of Inertia** of the body.

If m_1, m_2, m_3, ... be the masses of the infinitesimal portions which we may consider as particles A, B, C, . . ., the centre of inertia of the body will coincide with the centre of the system of parallel forces $m_1, m_2, m_3, \ldots$ acting respectively at the points A, B, C, . . . In finding therefore the centre of inertia of the body we proceed as in Article 57. Divide the line joining AB at a, so that

$$\mathrm{A}a : \mathrm{B}a = m_2 : m_1.$$

Next divide the line joining aC at b, so that

$$ab : b\mathrm{C} = m_3 : m_1 + m_2,$$

and so on. The point which we shall finally arrive at is the centre of inertia of the body.

From this construction it is evident that the centre of inertia of a body could be defined without reference to force, and that its position will depend only on the distribution of mass among the particles of the body.

68. The attractive force of the earth on any particle is a force along the line joining the particle with the earth's centre, and therefore the resultant action of gravity on any mass, that is, the weight of the mass, will be the resultant of a system of forces all passing through the centre of the earth. Now the distance of the centre of the earth is infinitely great compared with the dimensions of any body at the surface, and therefore the lines joining the centre of the earth with the particles of any body at the surface may, without appreciable error, be all considered as parallel. It follows that the action of gravity on the body may be considered as the resultant of a system of forces acting on the particles of the body parallel to the line joining any point of the body with the centre of the earth, each force being proportional to the mass of the particle on which it acts. Hence the resultant action of gravity on the body will be a single force, namely, the weight of the body, acting through a fixed point, the centre of inertia of the body. As a consequence of this assumption the position of the centre of gravity is always supposed to coincide with the position of the centre of inertia, and the two terms are used indifferently. The student must, however, note carefully that the ideas underlying the two terms are entirely different.

Example.—Three particles A, B, C, whose masses are 3, 4, 5 units respectively, lie in a straight line ABC, AB being 10 inches and BC 15 inches. Find the centre of inertia of the three masses.

The centre of inertia of the three masses will coincide with the point where the straight line ABC is cut by the resultant of three parallel forces of 3, 4, 5 units acting at A, B, C respectively. The resultant of these three forces would be a force of $3+4+5$, or 12 units, and if G be the point where the line of action of the resultant cuts AC,

$$12\ AG = 10 \times 4 + 25 \times 5$$

$$= 165,$$

from which

$$AG = 13\tfrac{3}{4} \text{ inches.}$$

Hence the centre of inertia of the three particles is at a point in AC distant $13\frac{3}{4}$ inches from A.

69. Centre of Inertia of Symmetrical Bodies.

In finding the centre of inertia of bodies we shall always suppose, unless it is explicitly stated to the contrary, that the bodies are of uniform density throughout.

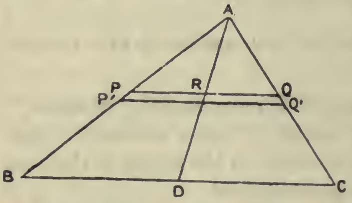
In the case where the body has a **Centre of Figure**, that is, where the body is symmetrical round some point, the centre of inertia coincides with the centre of figure. A sphere, an ellipsoid, and a cube are examples of symmetrical bodies, the centres of figure of the sphere and ellipsoid being the centres of the surfaces, and that of the cube the point of intersection of the diagonals. Hence the centre of inertia of a sphere or of an ellipsoid is the centre, and the centre of inertia of a cube is the point of intersection of the diagonals.

Similarly, the centre of inertia of a straight rod of uniform section is a point in that section which lies midway between the two ends. If we neglect the area of the section and look upon the rod as a straight line, we may say that the centre of inertia is the middle point of the rod.

In the case of a body which is in the form of a plate or lamina of uniform thickness, the centre of inertia will, of course, lie in a plane between the two bounding planes of the lamina. But if the thickness of the plate is so small that it may be neglected, we may look upon the plate as a surface, that is, as a body having length and breadth but not thickness, but at the same time differing from a surface in having mass, and having therefore a centre of inertia. If the surface has a centre of figure, that point will be the centre of inertia of the surface. Thus the centre of inertia of a circle is its centre, and the centre of inertia of any parallelogram, including rectangles and squares, is the point of intersection of the diagonals.

70. Centre of Inertia of a Triangle.

Let ABC be a triangular lamina whose thickness is so small that it may be neglected. Let D be the middle point of BC; then we shall prove that the centre of inertia of the triangle lies in AD.



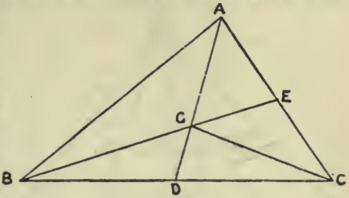
We may conceive the triangle to be made up of an infinite number of strips of infinitesimal thickness and breadth, all parallel to the base of the triangle. Let PQ Q'P' in the figure represent one of these strips, which, being of infinitesimal breadth, may be considered to be cut by the line AD in the point R. The centre of inertia of this strip may be considered to be the middle point of PQ, and we shall prove that R is that point. For by Euclid

$$\begin{aligned} PR : RA &= BD : DA \\ &= CD : DA, \text{ (since } BD = CD) \\ &= RQ : RA. \end{aligned}$$

Hence PR = RQ, that is, R is the middle point of PQ. Hence the centre of inertia of the strip PQ will lie in AD, and similarly with all the other strips. It follows that the centre of inertia of the triangle must lie in AD.

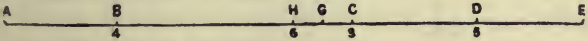
Similarly, if E be the middle point of the side CA, it may be shown in like manner that the centre of inertia lies in BE. Hence the centre of inertia must be G, the point where the two bisectors meet.

Since D is the middle point of BC, the area of the triangle ABD is equal to the area of the triangle ADC, and the area of the triangle BGD is equal to the area of the triangle GDC. Taking equals from equals we find the area of the triangle ABG equal to the area of the triangle AGC. Similarly we may prove that the area of BGC is equal to the area of ABG. Hence, since the three areas ABG, BGC, AGC are all equal, each is equal to one-third of the area of ABC, or two-thirds of the area of half ABC. Hence BGC is two-thirds of BEC, and therefore BG is two-thirds of BE. Similarly AG is two-thirds of AD. We have thus arrived at this result:—The centre of inertia of any triangle is a point on the line drawn from any angular point to bisect the opposite side at a distance from the angular point equal to two-thirds of the bisecting line.



Example 1.—Masses of 4, 3, 5 pounds are hung at points distant 1, 3, 4 feet respectively from the end of a lever 5 feet long and weighing 6 pounds. Find the centre of inertia of the lever and the masses.

Let A B H C D E be the lever, the mass of 4 pounds being hung on at B, the mass of 3 at C, and the mass of 5 at D. The mass of the lever, 6 pounds, may be supposed to be collected into H, the centre of inertia of the lever, and the middle point of the lever. We have therefore to find the centre of inertia of masses of 4, 6, 3, and 5 pounds respectively placed at B, H, C and D. Let G be the centre of inertia, then

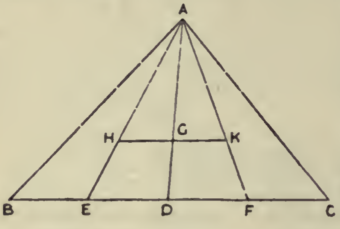


$$\begin{aligned}18AG &= 4 \times 1 + 6 \times 2\tfrac{1}{2} + 3 \times 3 + 5 \times 4\\ &= 4 + 15 + 9 + 20\\ &= 48,\end{aligned}$$

from which $$AG = \frac{48}{18} = 2\tfrac{2}{3} \text{ feet,}$$

which determines the position of G.

Example 2.—If D is the middle point of the side of a triangle ABC, show that the distance between the centres of inertia of the triangles ABD and ACD is one-third of BC.

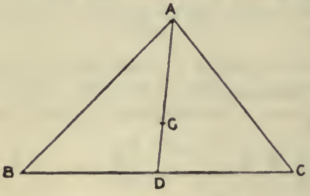


Let E and F be the middle points of BD and DC, and let G, H, K be the centres of inertia of the three triangles ABC, ABD, ACD respectively. Since G, H, K divide the lines DA, EA, FA in the same ratio, these three points lie in the same straight line, parallel to BC. Hence the triangles AHK and AEF are similar, and

$$HK : EF = AH : AE.$$

But $AH = \frac{2}{3} AE$, and therefore $HK = \frac{2}{3} EF$. But $EF = \frac{1}{2} BC$; hence HK, the distance between the centres of inertia of the triangles ABD and ADC, is $\frac{1}{3}$ BC.

Example 3.—Show that the centre of inertia of three particles of equal mass placed one at each angular point of a triangle coincides with the centre of inertia of the triangle.



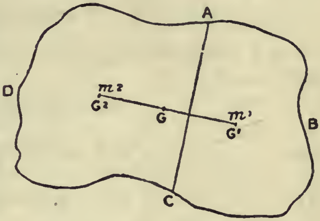
Let m be the mass of each particle. The centre of inertia of a mass m at B and a mass m at C is at D the middle point of BC. We may, therefore, replace the mass at B and the mass at C by a mass $2m$ at D. This mass $2m$ at D and the mass m at A will have for their centre of inertia a point G determined by the proportion

$$\begin{aligned} DG : GA &= m : 2m \\ &= 1 : 2. \end{aligned}$$

Hence DG is half of AG or a third of DA. Hence G coincides with the centre of inertia of the triangle ABC.

71. A body is made up of two parts, and the positions of the centres of inertia of the two parts are known, required to find the centre of inertia of the whole body.

Let a body ABCD be made up of two parts ABC and ADC, and let m_1, m_2 be the masses, and G_1, G_2 the positions of the centres of inertia respectively of the two portions ABC and ADC.



We may consider the masses m_1 and m_2 of the two parts to be collected at G_1 and G_2, and the problem reduces to that of finding the centre of inertia of two given masses, m_1 at G_1 and m_2 at G_2. If G be the centre of inertia of these two masses, G is the centre of inertia of ABCD.

Evidently G lies in the line G_1G_2, and its position will be found from the proportion

$$GG_1 : GG_2 = m_2 : m_1,$$

that is,

$$\frac{GG_1}{GG_2} = \frac{m_2}{m_1},$$

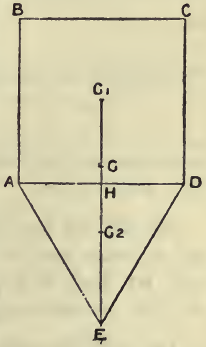
from which

$$\frac{GG_1}{GG_1 + GG_2} = \frac{m_2}{m_1 + m_2},$$

or

$$\frac{GG_1}{G_1G_2} = \frac{m_2}{m_1 + m_2}.$$

Now G_1G_2 is the distance between the centres of inertia of the two parts ABC and ADE of the body, and is therefore known. We can thus find from the last written equation the length GG_1, which is the distance of the centre of inertia of the whole body from the centre of inertia of the part ABC.



Example.—A square is described on the side of an equilateral triangle. Find the centre of gravity of the figure made up of the square and the triangle.

Let ABCD be the square and ADE the equilateral triangle. Then the centre of gravity of the square is the centre of figure G_1, and it is evident from the symmetry of the figure that G_1E cuts AD at right angles at its middle point H.

Hence the centre of gravity G_2 of the triangle ADE is in the line G_1E, its distance from H being given by the equation

$$HG_2 = \tfrac{1}{3} HE.$$

Let a denote the side of the square or the side of the triangle; then

$$HE = AE \sin 60^\circ = \frac{a\sqrt{3}}{2},$$

and
$$HG_2 = \tfrac{1}{3} HE = \frac{a}{2\sqrt{3}}.$$

Also
$$HG_1 = \tfrac{1}{2} \text{ side of square}$$
$$= \tfrac{1}{2} a;$$

therefore
$$G_1G_2 = \tfrac{1}{2} a + \frac{a}{2\sqrt{3}} = \frac{a(\sqrt{3}+1)}{2\sqrt{3}}.$$

The masses of the square and triangle are proportional to their areas, that is, are proportional to a^2 and $\frac{a^2\sqrt{3}}{4}$. Hence if G be the centre of gravity of the whole figure

$$GG_1 : GG_2 = \text{mass of triangle} : \text{mass of square}$$
$$= \frac{\sqrt{3}}{4} a^2 : a^2$$
$$= \sqrt{3} : 4.$$

Hence
$$\frac{GG_1}{GG_2} = \frac{\sqrt{3}}{4}, \text{ and } \frac{GG_1}{GG_1 + GG_2} = \frac{\sqrt{3}}{4+\sqrt{3}},$$

or
$$\frac{GG_1}{G_1G_2} = \frac{\sqrt{3}}{4+\sqrt{3}}.$$

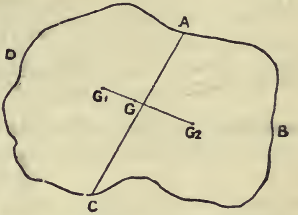
Substituting the value of G_1G_2, we get

$$GG_1 = \frac{a}{2} \cdot \frac{\sqrt{3}+1}{4+\sqrt{3}},$$

which determines the distance of C.G. of the whole figure from the C.G. of the square.

72. Given the positions of the centres of inertia of a body and of a part of the body which is cut off, to find the position of the centre of inertia of the remaining portion.

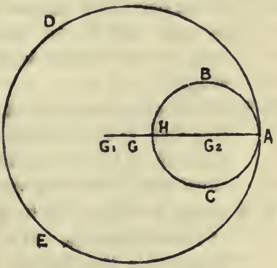
Let ABCD be the body of which the position of G, the C.I., is known, and let the portion ABC of which the position of the C.I., G_2, is also known be cut off. It is required to find the C.I., G_1, of the remaining portion.



Considering the body as made up of the two parts ABC and ADC, of which G_1 and G_2 are the centres of inertia, we have seen that the C.I. of the whole body, G, lies between G_1 and G_2. Hence if G and G_2 are known, it follows that G_1 lies in the straight line G_2G produced.

We find the position of G_1 by forming an equation expressing that the distance GG_1 bears to the distance GG_2 the inverse ratio of the remaining portion ADC to the portion ABC which is cut off.

Example.—A circular hole, 6 inches in diameter, is cut out of a circular disc of wood, 15 inches in diameter, close to the edge. Find the C.I. of the remaining portion.



The C.I. of the whole disc ADE before the hole is cut out is, of course, G, the centre of the disc. Let G_1 be the C.I. of the remaining portion, and G_2 the C.I. of the part ABC cut out. G_2 will evidently be the centre of the circle ABC. We may look upon the whole disc as made up of the part ABC cut out and of the remaining portion, the two centres of inertia being respectively G_2 and G_1, and the C.I. of the whole plate being G. Hence

$$GG_1 : GG_2 = \left\{ \begin{array}{l} \text{mass of the part cut off : mass} \\ \quad \text{of remaining part.} \end{array} \right.$$

$$= \left\{ \begin{array}{l} \text{area of part cut off : area of} \\ \quad \text{remaining part.} \end{array} \right.$$

Now the area of a circle varies as the square of the diameter. Hence the areas of the whole plate and of the part cut off are in the ratio of 15^2 to 6^2, or 225 to 36; hence the ratio of the part cut off to the remaining portion is 36 to 225 − 36, or 36 to 189;

hence $$GG_1 : GG_2 = 36 : 189.$$

Also $$GG_2 = GH + HG_2$$
$$= 1\tfrac{1}{2} + 3 = 4\tfrac{1}{2} \text{ inches.}$$

Hence $$\frac{GG_1}{4\frac{1}{2}} = \frac{36}{189} = \frac{4}{21};$$

therefore $$GG_1 = 4\tfrac{1}{2} \times \frac{4}{21}$$
$$= \frac{9}{2} \times \frac{4}{21} = \frac{6}{7} \text{ inch.}$$

EXAMPLES XVIII.

1. Two cubes of the same material, whose edges measure 4 inches and 5 inches respectively, are put together so that the middle point of a face of one coincides with the middle point of a face of the other. Find the C.G. of the combination.

2. A square and a rectangle of uniform material are joined together in one plane at a common side; find the length of the rectangle in order that the two may balance about that side, the thickness of the square being double the thickness of the rectangle.

3. The C.G. of the hull of a steamer is 100 feet from the stern-post. The C.G. of the engines is 90 feet from the stern-post. Find the C.G. of the whole, taking the weight of the hull to be four times that of the engines.

4. A square is divided by its diagonals into four triangles, and one of these is cut out. If a denote a side of the square, find, in terms of a, the distance of the C.G. of the remaining portion from the centre of the square.

5. On a straight line of length a a square is described and an isosceles triangle of equal area, the two figures being on opposite sides of the line. How far outside the square is the centre of inertia of the whole?

6. From an equilateral triangle whose side is a inches long, a triangular piece is cut off by a straight line drawn through the middle point of one side parallel to another side. Find the C.I. of the remaining portion.

7. A piece in the shape of a triangle is punched out of a circular disc,

the radius of which is 1 foot. Each side of the triangle is equal to the radius of the circle, and one angle of the triangle coincides with the centre of the circle. Find the C.G. of the remainder of the disc.

8. A hole, 1 foot square, is cut out of a board 4 feet square, the sides of the hole being parallel to the sides of the board and the middle point of one of them being at the centre of the board. Find the C.G. of the remaining part of the board.

9. Determine the perpendicular distance from the sides of the C.G. of a uniform triangular plate whose sides are 4 feet 4 inches, 4 feet 8 inches, and 5 feet.

10. The greatest angle of a given isosceles triangle is 120°; from the vertex and the middle points of the equal sides lines are drawn perpendicular to the base. Supposing the perimeter of the triangle and the perpendiculars to be uniform rods of the same material, show that the distance of the centre of gravity of this framework from the vertex is $a\,\frac{13+8\sqrt{3}}{24+8\sqrt{3}}$, where a = length of perpendicular from vertex.

11. Three squares of the same material and same uniform thickness whose areas are A, A, 2A, respectively are placed in one plane so that one corner of each square coincides with a corner of one of the other two, the three double corners thus being the angular points of a triangle whose sides are edges of the squares. Prove that the centre of gravity of the three squares lies in an edge of the largest square.

12. Two equal masses, each equal to m, are placed at the extremities A and B of the straight line AB. On AB a segment of a circle is constructed, and a third mass, also equal to m, moves along the circle. What is the locus of the centre of inertia of the three masses?

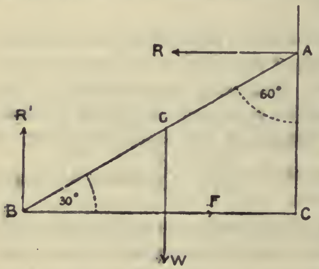
73. Statical problems involving the Force of Gravity.

In considering problems in statics which involve the force of gravity we assume that the action of gravity on a body is reducible to a single force, the weight of the body, and that this force acts vertically through the C.I. of the body.

EXAMPLE 1.—A uniform ladder whose weight is W rests with one end on a smooth horizontal plane, and the other against a smooth vertical wall, and is kept in equilibrium by a force F applied to the foot of the ladder in a direction perpendicular to the base of the wall. Find the pressures on the wall and the plane, and also the force F when the ladder is inclined to the vertical at an angle of 60°.

Let the figure represent a section through the ladder perpendicular to the wall, cutting the wall in the line AC.

Consider the forces acting on the ladder. First there is the action of gravity which reduces to a single force W, the weight of the ladder, acting through G, the middle point of AB. Next the ladder presses on the wall, and therefore by Newton's third law of motion the wall presses on the ladder with an equal force in the opposite direction. Since the wall is perfectly smooth, there is no friction (Chap. VI.) between the ladder and the wall, and therefore the mutual pressure between the wall and the ladder must be a force perpendicular to the wall. Hence the second force acting on the ladder is a force R acting horizontally at A.



Another force acting on the ladder is a force R′ at B, the pressure between the ground and the ladder, acting vertically. The remaining force is the force F acting at B horizontally. These four forces W, R, R′, F, keep the ladder in equilibrium, and it is required to find the three forces R, R′, F in terms of W. We express algebraically the conditions of equilibrium found in Article 64. These conditions will give three equations from which the three forces R, R′, F can be found in terms of W.

The vertical component of all the forces downwards is $W - R'$, and the horizontal component outwards from the wall is $R - F$. For equilibrium these components must both be zero. Hence

$$W - R' = 0, \text{ and } R - F = 0.$$

or $$R' = W, \quad R = F \text{ ....(1) and (2).}$$

Equation (1) gives the value of R′ in terms of W.

We find a third equation by expressing algebraically that the sum of the moments of all the forces round any point is zero. B will be the most convenient point to take, as two of the forces pass through B. The moment of W round B is $W \times BG \cos 30° =$

$W.\frac{AB}{2}.\frac{\sqrt{3}}{2}=\frac{W.AB\sqrt{3}}{4}$. The moment of R round B is $-R.AB\times$ $\cos 60°$ or $-\frac{R.AB}{2}$, the moment being negative because it is opposite in direction to the moment of W, which is considered positive. Hence since the algebraical sum of the moments must be zero,

$$W.\frac{AB\sqrt{3}}{4}-R.\frac{AB}{2}=0,$$

from which $$R=\frac{W\sqrt{3}}{2}\text{ ....(3).}$$

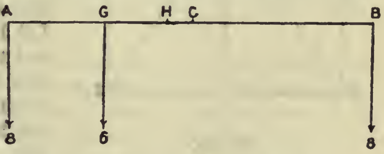
Hence from equations (2) and (3)

$$R=F=W\frac{\sqrt{3}}{2},$$

which gives R and F in terms of W.

Example 2.—Two weights of 8 lbs. each are hung one at each end of a straight lever 60 inches long and weighing 6 lbs. The lever, thus weighted, balances about a point 4 inches distant from its middle point. Find the C.G. of the lever.

Let G be the C.G. of the lever; then the lever is acted on by three parallel forces, the force 8 lbs. at A, the weight of the lever, 6 lbs., at G, and the force of 8 lbs. at B. These parallel forces are balanced by the pressure of the support on the lever at H, the point 4 inches distant from C the middle point, and therefore the system of forces 8, 6, 8 lbs. respectively at A, G, and B, and the pressure of the support at H, keep the lever in equilibrium. Hence the algebraical sum of the moments of these forces round any point in the plane is zero. Taking the point H, and expressing that the moments round H are equal, we get



$$6\times GH+8\times AH-8\times BH=0.$$

Let $AG=x$, then since H is 4 inches from the middle point C of the rod, $AH=26$ inches, $BH=34$ inches, $GH=AH-x=(26-x)$ inches. Substituting in the equation of moments, we get

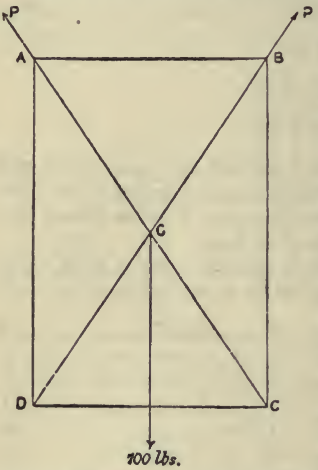
$$6\ (26-x)+8\times 26-8\times 34=0,$$

from which we find $$x=\frac{46}{3}\text{ inches.}$$

Hence the position of the C.G. is $\frac{46}{3}$ inches from the end A of the lever.

EXAMPLE 3.—The weight of a door is 100 lbs., its width is 4 feet, and the distance between the hinges 6 feet. What is the pressure on each hinge, assuming each bears half the weight of the door?

Let ABCD be the door hinged to the door-post at A and D. It is evident that the door will exert a thrust or pressure on the lower hinge D and a tension or pull on the upper hinge A. Hence by Newton's third of motion, the door will experience a pressure at the lower hinge and a tension at the upper hinge. By the supposition of the problem these two forces will be equal, and therefore their lines of action must be equally inclined to the vertical as the algebraical sum of their horizontal components is zero. Hence the angles ADB and CAD are equal. Let P, P be the pressure at D and the tension at A; then the door is in equilibrium under the action of the forces P,P, and the weight 100 lbs. acting at G the C.G. It follows that these three forces must meet in a point, and that point must be the C.G., since the angle ADB and CAD are equal. Expressing that the algebraical sum of the vertical components is zero, we get



$$P\cos ADB + P\cos CAD - 100 = 0,$$

or $$2\,P\cos ADB = 100.$$

Now $$\cos ADB = \frac{AD}{BD} = \frac{6}{\sqrt{(6^2+4^2)}} = \frac{6}{\sqrt{52}} = \frac{3}{\sqrt{13}}.$$

Hence $$2P \times \frac{3}{\sqrt{13}} = 100,$$

from which $$P = \frac{100\sqrt{13}}{6} = \frac{50\sqrt{13}}{3} \text{ lbs.}$$

EXAMPLES XIX.

1. A certain rod when unloaded balances about a point 4 feet from one end, its whole length being 9 feet. About what point will it balance when a weight double that of the rod is hung from the end just mentioned, and a weight triple that of the rod from the other end?

2. A straight lever which, when unweighted, balances about a point a third of its length from one end, balances about its centre when weighted with 7 lbs. at one end and 12 lbs. at the other. Find its weight.

3. A uniform lever weighing 1 lb. has a weight of 5 lbs. attached to one end and a weight of 10 lbs. at the other end. At what distance from its centre is the point about which it will balance?

4. A tapering rod balances about a point one-third of its length from the larger end. If a weight double its own weight be hung from the small end, about what point will it now balance?

5. A uniform rod rests with its ends on two props. At what point of the rod must a weight equal to the weight of the rod be hung to make the pressure on one prop double that on the other?

6. A uniform ladder rests in a vertical plane with one extremity on the ground, and the other supported by a vertical wall, both the ground and the wall being smooth. The lower end of the ladder is fastened by a string to a point on the ground, and the tension of the string is equal to half the weight of the ladder. Prove that the inclination of the ladder to the horizon is 45°.

7. A uniform rod weighs 12 oz. Find where it will balance when a weight of 6 oz. is hung on at one end and a weight of 16 oz. at the other.

8. A and B are two points in a horizontal line. A uniform rod whose length is AB and weight $20\sqrt{3}$ lbs. turns freely round a hinge at one end B. The other is fastened to a thread which passes over A, and carries a weight of 10 lbs. Find the inclination of the rod to the horizon when at rest (Art. 65).

9. ABC is a triangle, having C a right angle and BC longer than CA. Find the point in BC from which, when the triangle is suspended, the hypotenuse will be horizontal.

10. A square board ABCD is capable of turning freely in its plane, which is vertical, round A. It is held so that AB is vertical by a force acting along BC. Find the force along BC in terms of the weight of the board.

11. A straight uniform rod whose mass is 5 lbs. is attached to a hinge at one end and hangs in a vertical plane. A horizontal force of $\frac{5\sqrt{3}}{2}$ pound acts at its extremity. Show that when the rod is in equilibrium it is inclined to the vertical at an angle of 60°.

12. An isosceles triangle weighs 6 oz. Find the weight which must be placed at the vertex in order that the triangle may balance about the middle point of a perpendicular from the vertex on the base.

13. A sphere whose weight is 100 lbs. rests between two inclined planes which are inclined to the horizon at angles of 30° and 60°. Find the pressure exerted by the sphere on each of the planes.

14. A uniform beam rests with one end A on a smooth vertical wall; and the other end B is supported by a string fastened to a point C in the wall vertically above A. If the length of the beam be a feet, and the length of the string b feet, find the length of CA and the tension of the string.

15. A uniform beam of weight W, which can turn freely about a hinge at its lowest point, is held at an inclination α to the vertical by means of a string tied to its upper extremity, and fastened to a point vertically above the hinge and at a distance from it equal to the length of the beam. Show that the pressure on the hinge is equal to $W \cos \frac{\alpha}{2}$, and that its direction bisects the angle α.

16. Find the position of equilibrium of a rod resting on a smooth horizontal rail, its lower end pressing against a smooth vertical wall parallel to the rail. The length of the rod is 16 feet, and the perpendicular distance of the rail from the wall is 1 foot.

17. Four beams are jointed together so as to form a trapezoid; and the frame thus formed is set upright, with the base AB resting on a horizontal plane and DC parallel to AB. If a load of 12 cwt. is placed at each of the upper corners, D and C, find the stresses produced along the beams, the lengths being as follows: AB = 22 feet, DC = AD = BC = 12 feet.

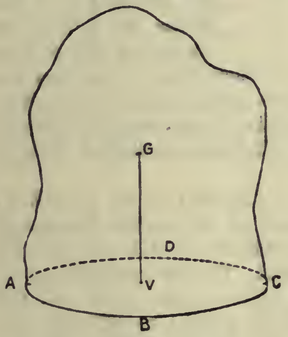
Point out which of the stresses are thrusts and which are tensions.

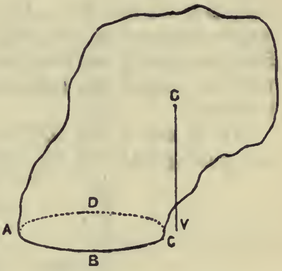
In this question the weights of the beams are, of course, to be neglected.

74. Cases of Equilibrium of Bodies under the Action of Gravity and of Fixed Supports.

When a body is supported from a fixed point and acted on by the force of gravity alone, the only condition of equilibrium is that the centre of gravity of the body should be vertically above or vertically under the point of support.

When a body is placed on a horizontal plane so that part of the surface of the body is in contact with the plane, as in the accompanying figures, the body will be in equilibrium if, as in the left-hand figure, the vertical line GV through the C.G. falls within the area of contact ABCD, and will topple over if, as in the right-hand figure, the vertical line GV through the C.G. falls without the area of contact ABCD.



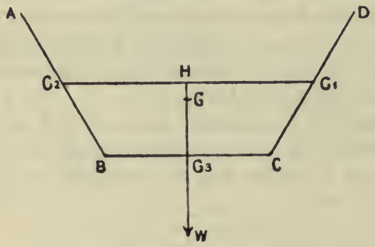


So also if a body be placed on a horizontal plane and be supported by three or more supports, *e.g.* a table supported on three legs, the body will stand or fall according as the vertical line through the C.G. falls within or without the area formed by joining the feet of the supports.

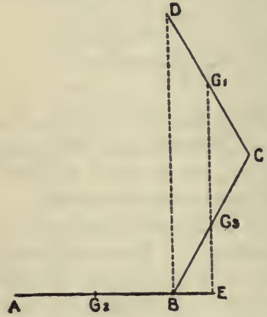
EXAMPLE 1.—Three uniform similar rods are fastened together so as to form a framework in the shape of half a regular hexagon. Show that the framework will rest in a vertical plane if one of the rods is horizontal.

Let AB, BC, CD be the three rods forming half a regular hexagon.

(1). If the middle rod BC is horizontal, the framework will rest in a vertical plane. The centres of gravity G_1 and G_2 of CD and AB are the middle points of CD and AB, and if G_1G_2 be joined and bisected at H, H will be the C.G. of the two rods AB and CD. Now H is vertically above G_3 the middle point of BC and the C.G. of BC. Hence the C.G. of the three rods will be some point G in the line HG_3. Hence the whole weight of the three rods may be supposed to be a single force acting vertically through G_3. Since the line of this force falls within the area which is in contact with the ground, the framework will be in equilibrium with BC horizontal.



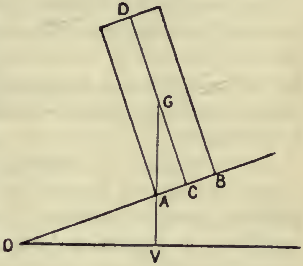
(2). If one of the end rods, as AB, rests on the ground, the line joining BD will, by a property of the regular hexagon, be perpendicular to AB, and therefore vertical. The line joining G_1 and G_3, the middle points of BC and CD, must be parallel to BD and must therefore also be vertical. Let G_1G_3 cut AB produced in E, and let W denote the weight of each rod. Then the weights of the two rods BC and DC will be equivalent to the weight 2 W acting vertically through E. Now,



$$BE = BG_3 \cos 60^\circ = \tfrac{1}{2} BC \cos 60^\circ$$
$$= \tfrac{1}{4} BC.$$

We have thus a force 2 W acting at E and a force W acting at G_2, both forces being vertical. Since $BG_2 = 2\,BE$, it follows that the resultant is a force 3 W acting vertically through B. Hence the resultant action of gravity on the three rods is a single force acting vertically through B, the extreme point in contact with the ground. The framework is therefore just in equilibrium.

EXAMPLE 2.—A cylinder whose height is 3 times the diameter of its base stands on a plane, horizontal at first, but afterwards inclined to the horizon at a gradually increasing inclination. Supposing friction between the cylinder and the plane sufficient to prevent sliding, find what will be the inclination of the plane to the horizon when the cylinder topples over.



Let the figure represent a section through the axis of the cylinder, just about to topple over, perpendicular to the inclined plane.

G, the C.G. of the cylinder, will be at the middle point of the axis CD, and in the position in which the cylinder is about to topple over the vertical GV will pass through A. Let α denote the inclination of the plane OACB to the horizon OV; then

$$\alpha = \text{angle AOV} = \text{angle AGC}.$$

Hence $$\tan\alpha = \frac{AC}{CG} = \frac{AB}{CD} = \frac{1}{3}.$$

From a table we find $$\tan 18^\circ\ 26' = \tfrac{1}{3}.$$

Hence the cylinder will be on the point of toppling over when the inclination of the plane to the horizon is 18° 26′.

75. Stable, Unstable, and Neutral Equilibrium.

If a body is in equilibrium in a certain position under the action of given forces, and if slightly disturbed from that position, it may behave in one of three different ways:—(1) It may return to the position from which it was disturbed; (2) It may move away still farther from its original position; (3) It may remain in equilibrium in the new position.

In the first case the body is said to be in **stable** equilibrium in the original position, in the second case in **unstable** equilibrium, and in the third case in **neutral** equilibrium.

A pendulum hanging at rest is an example of stable equilibrium. If slightly displaced in either direction in which it is free to move it will immediately return to its original position and oscillate about that position until it comes to rest. In general a body with one point fixed will rest in stable equilibrium if the C.G. lies below the fixed point.

An example of unstable equilibrium is seen in the case of a long rod balanced vertically on the hand. If slightly displaced the rod will tend to move away from the vertical position and fall to the ground. In general a body with one point fixed can rest only in unstable equilibrium when the fixed point is below the C.G.

As an example of neutral equilibrium take the case of a sphere resting on a level table. If slightly displaced from this position in any direction, it will remain at rest in its new position. In general a body will be in neutral equilibrium when the C.G. remains at the same height after as before being disturbed.

It is to be noticed that a body may be in one kind of equilibrium for a displacement in a certain direction and may be in another kind of equilibrium for a displacement in another direction. Thus, for example, a coin resting vertically on its edge is in neutral equilibrium for displacements in its own plane, while it is in unstable equilibrium for displacements in any other direction. Practically, however, the equilibrium would be considered very unstable.

A body is sometimes in equilibrium owing to a rapid motion of rotation. The explanation does not belong to elementary dynamics. An example of such stability is seen in the case of a bicycle, which at rest is practically unstable in an upright position, but which possesses a considerable degree of stability when driven along at a rapid rate. The same remark applies to a boy's top, which is practically in stable equilibrium when rotating rapidly. This kind of stability is called **Kinetic Stability.**

SECTION II.—KINETICS.

76. Acceleration of Gravity.

We saw in Article 38 that, when the resistance of the air was eliminated, all bodies at a given place moved under the action of gravity in vertical lines with a motion of acceleration which was the same for all.

We also saw that the acceleration due to gravity was different at different places on the earth's surface, and that its value over the United Kingdom does not differ much from 32·2 in British units, or 981 in C.G.S. units.

It is to be noticed that the value of the acceleration of gravity depends not only on the latitude but also on the height above the sea-level. The correction, however, for height is a very small one, and will be neglected in the remainder of this chapter.

We shall denote the acceleration of gravity by g, and we shall always assume that the motion takes place in vacuo, so that the resistance of the air need not be taken into account.

77. Bodies moving in a Vertical Line under the action of Gravity.

Problems involving the motion of bodies in vertical lines are solved on the assumption of a uniform acceleration g in the vertical direction. Such problems will be solved by the formulæ of Article 27, if we substitute g for a in the equations of that Article.

We shall divide the problems which may be proposed regarding the motion of bodies in vertical lines under the action of gravity into three classes:—(1) Problems in which bodies are projected vertically downwards; (2) Problems in which bodies are let fall from rest; (3) Problems in which bodies are projected vertically upwards.

I. Bodies Projected Vertically Downwards.

We use the formulæ of Article 27, changing a into g. Thus if u denote the velocity with which the body is projected downwards, s the space through which it falls in the time t, and v the velocity at the end of that time, then

$$v = u + gt \quad \text{........................} (1).$$
$$s = ut + \tfrac{1}{2} gt^2 \quad \text{......................} (2).$$
$$v^2 = u^2 + 2\,gs \quad \text{......................} (3).$$

EXAMPLE.—A body is projected downwards with a velocity of 10 feet per second. How far will it fall in the next 3 seconds, and what will be its velocity at the end of that time?

Here $u=10$, $t=3$, and $g=32{\cdot}2$;
hence s, the space fallen through in 3 seconds, is

$$= 10 \times 3 + \tfrac{1}{2} \times 32{\cdot}2 \times 9$$
$$= 174{\cdot}9 \text{ feet.}$$

Also v, the velocity at the end of 3 seconds, is

$$= 10 + 3 \times 32{\cdot}2 = 106{\cdot}6 \text{ feet per second.}$$

II. Bodies Falling from Rest.

In this case $u=0$, and we therefore use the formulæ of the first special case referred to in Article 27, changing a into g—

$$v = gt \ (1), \quad s = \tfrac{1}{2} gt^2 \ (2), \quad v^2 = 2gs \ (3).$$

EXAMPLE 1.—A body falls from rest under gravity. After falling through 100 feet, it has half of its velocity destroyed. How far will it fall in the next 2 seconds?

Let v denote the velocity of the body after falling through the hundred feet; then

$$v^2 = 2 \times 32{\cdot}2 \times 100 = 6440,$$

from which $v = 80{\cdot}2$ feet per second approximately.

Now we are asked to find how far the body will fall in the next 2 seconds. Since its velocity after falling 100 feet is 80·2 feet per second, and since half the velocity is then destroyed, the problem reduces to that of finding how far a body projected downwards with a velocity of 40·1 feet per second will fall in 2 seconds. This problem will be solved by the formula (2) of the first case. If s denote the space, then

$$s = 40{\cdot}1 \times 2 + \tfrac{1}{2} \times 32{\cdot}2 \times 2^2$$
$$= 144{\cdot}6 \text{ feet.}$$

EXAMPLE 2.—A body falls from rest. Find the space described in the n^{th} second of its fall.

The space described in n seconds is $\frac{1}{2} gn^2$; and the space described in a time which is one second shorter, that is, in $(n-1)$ seconds, is $\frac{1}{2}g(n-1)^2$. The space described in the n^{th} second is evidently

$$= \text{space described in } n \text{ sec.} - \text{space described in } (n-1) \text{ sec.}$$
$$= \tfrac{1}{2}gn^2 - \tfrac{1}{2}g(n-1)^2$$
$$= \tfrac{1}{2}g(2n-1).$$

III. Bodies Projected Vertically Upwards.

If a body is projected vertically upwards with a velocity u, the body has an acceleration g in the direction opposite to that in which the space s is measured, and we therefore use the formulæ of the second special case of Article 27, changing a into $-g$—

$$v = u - gt \quad \text{..........} \quad (1).$$
$$s = ut - \tfrac{1}{2}gt^2 \quad \text{..........} \quad (2).$$
$$v^2 = u^2 - 2gs \quad \text{..........} \quad (3).$$

To find how high a body will rise when projected vertically upwards with a given velocity u, and the time it takes to rise to its highest point, we put $v=0$ in these three formulæ, and the corresponding values of s and t will give the height and time required.

From formula (3) we obtain, on putting $v=0$,

$$0 = u^2 - 2gs,$$

and this gives $$s = \frac{u^2}{2g},$$

the height to which the body will rise.

From formula (1) we obtain, on putting $v=0$,

$$0 = u - gt,$$

and this gives $$t = \frac{u}{g},$$

the time the body will take to rise to its highest point.

Example 1.—A body is thrown vertically upwards with a velocity of 200 feet per second. How long will it go on rising, and what height will it reach? Also what will be its velocity, and where will it be after 2, 5, and 8 seconds respectively?

Here $u=200$, and we shall take $g=32$. By the formula the time the body takes to rise to its highest point is

$$\frac{u}{g} = \frac{200}{32} = 6{\cdot}25 \text{ seconds.}$$

Also the height to which it will rise is

$$\frac{u^2}{2g}=\frac{200^2}{2\times 32}=625 \text{ feet.}$$

The body therefore will continue to rise for 6·25 seconds, and will attain a height of 625 feet.

To find the velocity at a given time we use formula (1). Hence

$$\begin{aligned}\text{velocity after 2 seconds} &= 200-2\times 32\\ &= 136 \text{ feet per second;}\\ \text{velocity after 5 seconds} &= 200-5\times 32\\ &= 40 \text{ feet per second;}\\ \text{velocity after 8 seconds} &= 200-8\times 32\\ &= -56 \text{ feet per second.}\end{aligned}$$

This last result, being negative, shows that at the end of 8 seconds from the moment of projection the body has already reached its highest point, and has fallen so long as to attain a velocity of 56 feet per second downwards.

To find the height at a given time we use formula (2). Hence

$$\begin{aligned}\text{height at the end of 2 seconds} &= 200\times 2-\tfrac{1}{2}\times 32\times 2^2\\ &= 400-64\\ &= 336 \text{ feet;}\\ \text{height at the end of 5 seconds} &= 200\times 5-\tfrac{1}{2}\times 32\times 5^2\\ &= 600 \text{ feet;}\\ \text{height at the end of 8 seconds} &= 200\times 8-\tfrac{1}{2}\times 32\times 8^2\\ &= 576 \text{ feet.}\end{aligned}$$

EXAMPLE 2.—With what velocity must a stone be thrown vertically upwards to reach a height of 182 yards?

Let u be the velocity in feet per second; then since 182 yards = 546 feet, and since a body projected upwards with velocity u rises to a height $\frac{u^2}{2g}$, we get the equation

$$\frac{u^2}{2g}=546,$$

or

$$\begin{aligned}u^2 &= 2g\times 546=2\times 32\times 546\\ &= 34944;\end{aligned}$$

taking the square root, we get $u=186{\cdot}9$ feet per second. The body must therefore be projected upwards with a velocity of 186·9 feet per second.

EXAMPLES XX.

[In this set of Examples it is assumed that $g=32$ in British units and $=981$ in C.G.S. units.]

1. How far will a body fall during the time in which its velocity increases from 40·25 feet per second to 88·55 feet per second?

2. A stone is thrown vertically upwards with a velocity sufficient to carry it to a height of 100 feet. Find its velocity and height after $1\frac{1}{4}$ seconds.

3. St. Rollox chimney-stalk at Glasgow is 445 feet high. (1) At what rate must a bullet be shot vertically upwards from the ground to reach the top in one second? (2) What would be its rate in passing the top? (3) How high would it rise? (4) After what time would it return to the ground?

4. A body is thrown vertically upwards with a velocity of 110 feet per second. After how many seconds will it be moving downwards with a velocity of 66 feet per second?

5. A body is allowed to fall from rest under the action of gravity for 2 seconds. If the action of gravity were then to cease for 6 seconds, and then to act for a further period of 2 seconds, find the total space that the body would describe.

6. A stone after falling from rest for one second strikes a pane of glass, in breaking through which it loses half its velocity. How far will it fall in the next second?

7. A body is projected upwards with a velocity of 10 metres per second. How high will it rise, and after what time will it return to the ground?

8. How long will a body take to fall from rest through 1588 centimetres, and what will be its velocity at the end of that time?

9. A bullet is shot vertically upwards with a velocity of 16,400 centimetres per second. Find the spaces traversed by it in the twelfth and thirtieth seconds afterwards.

10. A body is thrown vertically upwards with a velocity of 124 feet per second, and after an interval of 2 seconds another body is thrown upwards with the same velocity. Find the height above the ground and the velocity of the two bodies when they meet.

11. If two bodies be dropped from the same point, but not at the

same time, show that the distance between them increases uniformly with the time.

12. A, B, C, D are points in a vertical line, the lengths of AB, BC, CD being equal. If a body falls from rest at A, prove that the times of describing the distances AB, BC, CD are as

$$1 : \sqrt{2}-1 : \sqrt{3}-\sqrt{2}.$$

13. Two bodies are dropped from the same point, one of them at an interval of 3 seconds after the other. Find the distance between them after a further interval of 3 seconds.

14. A body descends 1472·5 centimetres while its velocity increases from 981 centimetres to 1962 centimetres per second. What is the value of g in C.G.S. units at the place of observation?

15. Find the value of g in British units at a place where a body falls 2·977 feet in the first ·43 second.

16. A body falling from rest under the action of gravity describes one half of its entire space during the last second. Find the whole space described and the time of falling in feet and seconds.

17. In the nth second of its motion a falling body describes h feet. What was its initial velocity?

18. A body which has fallen vertically comes to the ground with a velocity v. How long before the impact was it h feet from the ground?

19. A body under the action of a constant force traverses in the tenth second from rest twice the distance it would have traversed in the fifth second from rest under the action of gravity. Compare the forces.

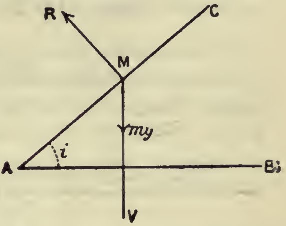
78. Other cases of Bodies moving under the Action of Gravity with Uniform Acceleration.

Besides the case of motion in a vertical line there are cases in which a body moves with uniform acceleration under some constraint. We shall consider three such cases.

I. Motion on an inclined plane.

Let M be a body, considered as a particle, mass m, on the inclined plane AC; and let i denote the angle CAB, the inclination of the inclined plane to the horizon. The body

M is acted on by two forces only, the reaction R of the plane and the weight mg vertically downwards. Since the force R acts perpendicular to the plane it has no component along the plane, and therefore can have no effect in causing motion along the plane. The force then which causes the motion down the plane is the component down the plane of the vertical force mg. This component is



$$mg \cos \text{VMA, or } mg \sin \text{MAB, or } mg \sin i.$$

The acceleration produced down the inclined plane by this force acting on the mass m is, by Newton's second law of motion,

$$\frac{mg \sin i}{m} \text{ or } g \sin i.$$

If we put $i=0$, that is, if we suppose the body on a horizontal plane, we have $\sin i=0$, and therefore the acceleration is zero, as we should expect.

If we put $i=90^\circ$, that is, if we suppose AC is vertical, we get $\sin i=1$, and the acceleration is g. It is evident that this brings us back to the case of a body falling vertically under no constraint.

Example.—A body is thrown with a velocity of 162 feet per second up an inclined plane which makes an angle of 30° with the horizon. How soon will it come to rest, and how far from the point of projection?

We use the formulæ of Case III. of the preceding Article with this modification, that we substitute $g \sin 30^\circ$ for g. Now $\sin 30^\circ = \frac{1}{2}$, and therefore $g \sin 30^\circ = \frac{1}{2} g$. Hence the time and distance required are

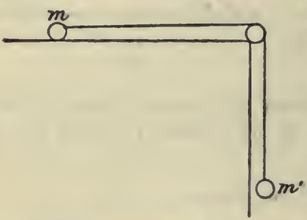
$$\frac{u}{\frac{1}{2} g} \text{ and } \frac{u^2}{2 \times \frac{1}{2} g} \text{ respectively.}$$

Now $u=162$, and $g=32$; hence the time and distance are

$$\frac{162}{16} = 10{\cdot}125 \text{ seconds, and } \frac{162^2}{32} = 820 \text{ feet approximately.}$$

II. A body of mass m lies on a horizontal table and is attached by means of a string which passes over a pulley at the edge of the table to another mass m' which hangs vertically.

In considering this arrangement we shall assume that the tensions in the horizontal and vertical parts of the string are equal. This is equivalent to neglecting the weight and imperfect flexibility of the string, and the inertia and friction of the pulley. This tension is the only force acting horizontally on m (friction between m and the table being neglected), and therefore m will move along the table in the straight line joining its initial position with the pulley. The mass m' will fall vertically.



Since the string will remain stretched during the motion, the two masses m and m' will at any time during the motion be moving with the same velocity and the same acceleration. Let α denote the acceleration of either mass at any point of time during the motion, and let T denote the tension of the string at the same time.

Consider the motion of the mass m'. It is acted on by two forces, one, its weight $m'g$, vertically downwards, the other, the tension T, vertically upwards. The resultant of those two forces is $m'g - \mathrm{T}$ vertically downwards. Under the action of this resultant force the mass m' is falling vertically with an acceleration α. Hence by Newton's second law of motion we get as the *equation of motion* for m'

$$m'\alpha = m'g - \mathrm{T} \quad\text{..........(1)}.$$

Similarly, the equation of motion for m is

$$m\alpha = \mathrm{T} \quad\text{..........(2)}.$$

By adding equations (1) and (2) we get

$$\alpha(m + m') = m'g,$$

giving

$$\alpha = m'g \,/\, (m + m') \quad\text{..........(3)}.$$

Substituting this value of a in equation (2), we obtain

$$\begin{aligned} T &= ma \\ &= mm'g \big/ (m+m') \text{..........(4).} \end{aligned}$$

T is, of course, in absolute units.

From (3) and (4) it follows that the acceleration and the tension are both constant throughout the motion. The motion is thus an example of uniform acceleration, and is completely determined by (3) and (4).

EXAMPLE 1.—If $m=14$ lbs., $m'=2$ lbs., find the space through which m' falls in 3 seconds, the common velocity at the end of that time, and the tension of the string. Take $g=32$.

By (3) the acceleration is

$$\frac{2g}{2+14}=\frac{64}{16}=4 \text{ feet per second per second.}$$

Hence by Article 27 the space through which m' falls in 3 seconds is

$$\tfrac{1}{2}\times 4\times 3^2=18 \text{ feet;}$$

and the common velocity at end of that time is

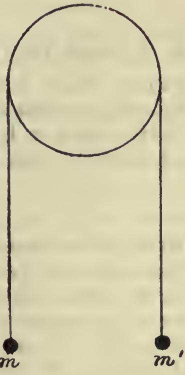
$$3\times 4=12 \text{ feet per second.}$$

The tension of the string is by (4)

$$\frac{2\times 14\times g}{2+14}=\frac{28\times 32}{16}=56 \text{ poundals.}$$

EXAMPLE 2. If the string is 12 feet long, and if m' start from the position of the pulley, show that the mass m will leave the table in 2 seconds, if $m' : m=3:13$.

III. Motion of two masses m and m' connected by a string passing over a smooth pulley.



If the masses m and m' are equal, it is evident that there will be a balance, and that the masses will remain at rest. But if one mass (m suppose) is greater than the other m', then m will begin to descend and m' to ascend, and it is evident that at each instant the velocity and acceleration of m downwards will be equal respectively to the velocity and acceleration of m' upwards.

Using a and T for the acceleration and tension at any instant during the motion, we have as the *equations of motion*—

for m, $ma = mg - T$ ....(1).

for m', $m'a = T - m'g$ ....(2).

Adding (1) and (2) we get

$$a(m+m') = g(m-m'),$$

or $a = g(m-m') \big/ (m+m')$ ....(3).

Multiply (1) by m' and (2) by m; then by subtracting we get

$$0 = 2mm'g - T(m+m'),$$

from which $T = 2mm'g \big/ (m+m')$ ....(4).

Results (3) and (4) show that the acceleration and the tension are both constant throughout the motion.

Example.—To one end of a string hanging over a pulley a mass of 5 oz. is attached, and to the other end masses of 3 oz. and 4 oz. respectively. The masses are allowed to move for 3 seconds when the 4 oz. is removed. Find how long and how far the 5 oz. will continue to ascend.

The end to which the 3 and the 4 ounces are attached will descend and the other end will ascend. By the formula the acceleration is (m being $3+4$ or 7 and m' being 5)

$$\frac{7-5}{7+5} \times g = \frac{g}{6} = \frac{16}{3}.$$

In 3 seconds the velocity acquired will be $\frac{16}{3} \times 3$ or 16 feet per second. Hence at the time when the 4 oz. is removed the 5 oz. is moving upwards with a velocity of 16 feet per second.

After the 4 oz. is removed, the 5 oz. will have a downward acceleration the value of which is

$$\frac{5-3}{5+3} g = \frac{g}{4} = 8.$$

Hence the acceleration downwards of the 5 oz. in the second part of the motion is 8 feet per second per second. The question now is: In what time will a body moving with a velocity of 16 feet per second be brought to rest if it is moving with a negative acceleration of 8: We put $u = 16$, $a = 8$ in the first formula of Case II. of Article 27. We thus find the time t by the equation

$$0 = 16 - 8t,$$

from which $t = 2$ seconds.

Also, to find the space through which the 5 oz. will continue to ascend, we put $v=0$, $u=16$, and $a=8$ in the second formula of Case II. of Art. 27. We thus obtain $0=16^2-2\times 8\times s$, from which we find $s=16$. Hence the 5 oz. will continue to ascend for 2 seconds and for a distance of 16 feet after the 4 oz. has been removed.

79. Attwood's Machine.

If in the last case of motion considered in the preceding Article the difference between the mass m and the mass m' is very small compared with either, then $m-m'$ will be small compared with $m+m'$, the fraction

$$\frac{m-m'}{m+m'}$$

will be very small, and the acceleration will only be a small fraction of g. The motion therefore will be very slow, and there will be no difficulty in observing with considerable accuracy the distance through which the mass m' will fall in a given time. From such observations a rough determination could be made of the value of g at any place on the earth's surface. Knowing the time and the distance we can find at once the value of the acceleration, and since the acceleration is a known fraction of g we can immediately determine the value of g. A value of g obtained in this way would, however, be only a first approximation to the true value, as no allowance is made for the friction of the axis of the pulley on its bearings and for the inertia of the pulley.

In **Attwood's Machine** the friction is almost altogether got rid of by making the pulley rest on wheels known as friction wheels instead of on fixed bearings. To each end of the string passing over the pulley boxes of equal mass are attached, and the motion is caused by placing on one of the boxes a small bar of metal whose mass is small compared with that of the boxes. The inertia of the pulley, that is, the resistance of the pulley to be set in rotation, is generally allowed for by supposing the mass of each box to be greater than it really is.

In making an experiment with Attwood's machine the small bar of metal is placed on one of the boxes, when motion immediately ensues, the box on which the bar lies descending and the other ascending. In its descent the former passes through an opening in the form of a ring, large enough to allow the box to pass freely through, but not large enough to admit of the passage of the bar. The result is that the bar remains behind supported by the edges of the opening, and consequently from the time of passing through the opening the descending box has no acceleration. It will, therefore, continue to descend with the velocity it had at the moment it passed through the opening, and we find that velocity by observing how far it descends in one second after passing through the opening. We have, then, all the data for determining the acceleration of gravity.

Let m denote the mass of each box, including the allowance for the inertia of the pulley, and let m' denote the mass of the bar of metal. Then the acceleration of the boxes, one downwards and the other upwards, when the bar is placed on one of the boxes is

$$\frac{m'}{2m+m'}g.$$

The distance (say s) from the point where the box began to descend to the opening is measured, and the velocity (say v) while passing through the opening is determined, and therefore a body with the above acceleration gains a velocity v in passing over a distance s; hence

$$v^2 = 2\frac{m'}{2m+m'}g\,s.$$

The values of v, s, m, and m' being all known, this equation will give the value of g.

Example.—In an Attwood's machine the equal masses are 94 lbs. and the additional mass is 2 lbs. The ring for intercepting the additional mass is placed at a distance of 1 foot 8 inches below the point from which the descending mass begins to move, and that mass descends 12 feet 9 inches in the next 12 seconds after passing through the ring. Find the acceleration of gravity at the place where the experiment was made.

The acceleration in this case is

$$= \frac{2}{2\times 94+2}g = \frac{g}{95}.$$

The velocity of the descending mass after falling 1 foot 8 inches with this acceleration is

$$=\sqrt{2\cdot\frac{g}{95}\cdot 1\tfrac{2}{3}}=\sqrt{\frac{2g}{57}}.$$

This is the velocity of that mass at the moment of passing through the ring. Now, after passing through the ring the descending mass falls 12 feet 9 inches in 12 seconds, equivalent to a velocity of

$$\frac{12\tfrac{9}{12}}{12}\text{ feet per second.}$$

Hence

$$\sqrt{\frac{2g}{57}}=\frac{12\tfrac{9}{12}}{12}=\frac{51}{48}=\frac{17}{16}.$$

Squaring both sides of this equation, we get

$$\frac{2g}{57}=\left(\frac{17}{16}\right)^2,$$

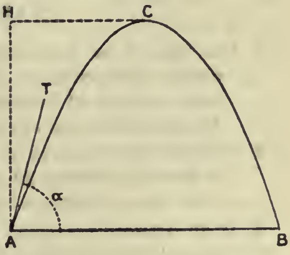
from which we find

$$g=32{\cdot}17\text{ nearly.}$$

80. Body Projected in any other Direction than the Vertical.

If a body is projected in any direction not vertical, it can be proved that, if the resistance of the air were neglected, the curve which the body would describe would be a curve ACB shown in the diagram, and known as a parabola.

Let the body be projected from the point A with velocity u in the direction AT, AT making an angle α with the horizontal line AB. Then without assuming any property of the parabola we can find the height AH to which the body rises, the time the body remains in the air, that is, the time the body takes to go from A to B, and the distance AB, known as the range.



The velocity of projection u can be resolved into the horizontal velocity $u\cos\alpha$ and the vertical velocity $u\sin\alpha$. Since the only force acting on the body is the force of gravity vertically downwards, the body throughout the motion will preserve its horizontal velocity unchanged. Consider the vertical motion apart from the horizontal motion. At the point of projection the vertical velocity is $u \sin \alpha$, and by Article 77 a body with this vertical velocity will rise to a height $\frac{u^2 \sin^2 \alpha}{2g}$ in a time $\frac{u \sin \alpha}{g}$. Hence the height AH of the highest point C is $\frac{u^2 \sin^2 \alpha}{2g}$, and the time of rising through the arc AC is $\frac{u \sin \alpha}{g}$. Similarly the time of falling through the arc CB will be $\frac{u \sin \alpha}{g}$, and therefore the whole time of flight is $\frac{2u \sin \alpha}{g}$. During this time the body is moving horizontally with the constant velocity $u \cos \alpha$, and therefore the range AB is $\frac{u \cos \alpha \times 2u \sin \alpha}{g}$ $= \frac{2u^2 \sin \alpha \cos \alpha}{g}$.

Example.—A rifle is discharged horizontally 1 foot above the surface of a lake, and the bullet strikes the water 220 feet off. What is its velocity?

Let u be the velocity of projection; then while the bullet is falling through 1 foot under the action of gravity it moves 220 with the constant velocity u. The time of falling 1 foot under the action of gravity is (taking $g = 32$) $\frac{1}{4}$ second. Hence with the velocity u the body moves 220 feet in $\frac{1}{4}$ second, or 880 feet in one second. Hence

$$u = 880 \text{ feet per second.}$$

EXAMPLES XXI.

1. One mass in an Attwood's machine is 500 grammes, and the other is 460 grammes. In two seconds from rest the greater mass descends 81·8 centimetres. What value would this give for the acceleration of gravity?
2. If one of the masses in an Attwood's machine be 33 lbs., and if it descends through 50 feet from rest in 10 seconds, find the other mass. Take $g = 32$.
3. In an Attwood's machine the masses are 104 grammes and 114 grammes respectively. Neglecting friction and the inertia of the thread and pulley, determine the tension of the thread in dynes and the acceleration of the mass. Take $g = 981$.
4. A mass of 2 lbs. hangs over the edge of a smooth table, and

draws a mass of 3 lbs. laid on the table. Find the space described and the velocity acquired in a quarter of a second.

5. At the extremities of a string which passes over a frictionless pulley two masses m and m' are attached. Initially the masses are at the same height above the ground, and 3 seconds after liberation the one is 72 feet above the other. Find the ratio of m to m'.

6. Given the height and length of a smooth inclined plane, find the acceleration of the velocity of a body which slides down it under the action of gravity.

   If the plane has a height of 3 feet for every 10 feet of its length, what distance would a particle, descending along it from a state of rest, describe in $2\frac{1}{2}$ seconds?

7. Find how soon a mass will slide down a slope 100 feet long, perfectly smooth, inclined to the horizon at 30°.

8. There is a table, 8 feet long, on which is placed a mass of 10 lbs. This is attached to a cord which runs over two pulleys, one at each end of the table. To one end of the cord is attached a mass of 5 lbs., and to the other a mass of 3 lbs. The system starts from rest with the 10-lb. mass at the side of the table nearest the 3-lb. mass. Neglecting friction and the inertia of the cord and pulleys, find the acceleration, tensions of the cord in the two parts, and the time the 10-lb. mass takes to cross the table.

9. A ball rolling down an inclined plane describes $2\frac{1}{2}$ feet in the first second of its motion. What distance does it describe in the third second?

10. If a body falls from rest down the height of an inclined plane in half the time it would take to slide down its length, what is the inclination of the plane to the horizon?

11. A mass of $6\frac{3}{4}$ lbs. is placed on a horizontal table and connected by a string to a mass of $1\frac{1}{4}$ lbs. which hangs over the edge of the table. If the latter body is allowed to fall dragging the former after it, what force does it exert on the former body and what is the accelerative effect on the velocity of that body?

12. A is the lowest point of the vertical diameter BA of a circle whose centre is C. Take P any point in the lower half of the circumference and draw a tangent PT to meet BA produced in T. Show that the time in which the body would slide down the chord PA is to the time in which it would slide down PT as

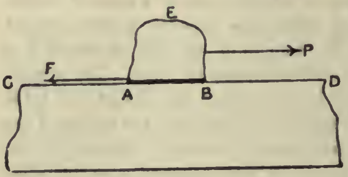
$$\sqrt{AB} : \sqrt{CT}.$$

13. A particle is projected horizontally, with a velocity of 32 feet per second, from a point 128 feet above the ground. Find the magnitude of the velocity and the tangent of the angle its direction makes with the vertical when the particle has fallen half-way to the ground, taking $g=32$.

CHAPTER VI.—FRICTION.

81. Definition of Friction.

Friction is the resistance which we experience when we attempt to make one body slide over another.



The direction in which the force or resistance of friction acts is the direction of tangent to the surfaces in contact. Thus, if a body ABE is at rest on a plane surface CD, and we attempt to drag it along the surface by a force P, the motion will be resisted by the force of friction F, which acts along the surface CD in the direction opposite to that in which the force P tends to move the body.

It is a matter of common observation that the amount of the force of friction between two bodies depends on the nature of the surfaces which are in contact. Thus it requires less force to drag a body such as a curling-stone along ice than along the ground, that is, there is less friction between the curling-stone and the ice than between the curling-stone and the ground.

When a body lies at rest on a horizontal table, no friction is called into account, as there is no force tending to make the body slide along the surface of the table. If we now apply a force to the body tending to drag it along the table, the force of friction begins to act and tends to prevent the body from being set in motion, just as much friction being called into action as to prevent motion. If the force tending to move the body along the table be gradually increased, we find that for a certain value of the force the force of friction is about to be overcome and the body is on the point of motion. From this we infer that the force of friction between two bodies is limited in amount, and that the maximum amount of friction is called into action when one body is on the point of sliding over the other. When we speak hereafter of the friction between two bodies, we shall always mean the maximum amount of friction.

Friction between surfaces is greatly diminished by the use of unguents. It is for the purpose of diminishing the friction between the axles and their bearings that oil and grease is used in machinery, and it is for the same purpose that in launching a ship the ways—the supports on which the vessel slides down into the water—are well smeared with grease.

82. Laws of Friction.

The following laws regarding the friction of two surfaces in contact were discovered by Coulomb (1781) and confirmed by Morin (1832), both officers in the French army:—

> (1) The friction depends on the nature of the surfaces in contact, but is independent of the extent of the areas in contact as long as the pressure between the two surfaces is the same.
>
> (2) The friction between two given surfaces is proportional to the pressure between them.

These laws, which were given by Coulomb for the friction between two surfaces on the point of sliding over each other, were afterwards found to be true by Morin for two surfaces actually sliding over each other. The friction between two surfaces sliding over each other is called **kinetic friction.** Morin found that the kinetic friction between two surfaces is less than the statical friction, and that it was independent of the velocity with which one surface slides over the other.

The resistance experienced when one surface rolls over another is called **rolling friction.** It appears from experiment that rolling friction is less than kinetic friction. Advantage is taken of this, as every one knows, in the use of wheels in carriages. The resistance to the rolling of a wheel on the ground has been found to vary as the weight, and inversely as the diameter.

83. Coefficient of Friction.

We have seen that the force of friction between two surfaces is proportional to the pressure. It follows from algebra that the fraction obtained by dividing the friction by the pressure will always be the same for the same two surfaces. This fraction is called the coefficient of friction.

Let R be the pressure between the two surfaces, F the friction, and μ the coefficient of friction; then

$$\frac{F}{R}=\mu,$$

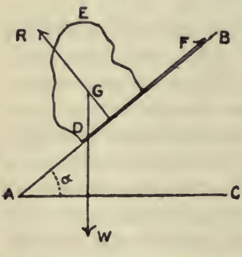
from which $$F=\mu R.$$

Hence if we know the pressure between two surfaces and the coefficient of friction for these two surfaces, we can find immediately the friction between the two surfaces.

84. Method of Determining the Coefficient of Friction between Two Surfaces.

The coefficient of friction between two surfaces is found by experiment. An apparatus, of which the figure represents a section, is constructed.

AB is the section of a plane hinged at A and made of one of the materials the friction between which is to be found. AC is the horizontal line in the plane of the section. DEG is a body of the other material resting with a plane surface in contact with the plane AB.



The plane of which AB is the section is gradually raised from the horizontal position AC until it comes into a position in which the body DEG is just about to slide down the plane. In this position the maximum amount of friction between the two bodies is called into account. Let us suppose that the figure represents this position, α being the angle which AB makes with the horizontal line AC.

The body DEG is in equilibrium, and therefore (Article 53) the algebraical sums of the components of all the forces acting on DEG along and perpendicular to the plane AB must each be zero.

Let R be the reaction of the plane, that is the force with which the plane AB presses on the body DEG, let F be the friction acting along AB tending to prevent motion down the plane, and let W be the weight of the body DEG. The sum of the components down the plane is $W\sin\alpha - F$, and this is zero. Hence

$$F = W\sin\alpha \qquad \text{..................(1)}$$

The sum of the components perpendicular to the plane is $R - W\cos\alpha$, and this is zero. Hence

$$R = W\cos\alpha \qquad \text{........................(2)}$$

Dividing equation (1) by equation (2) we obtain

$$\frac{F}{R} = \frac{W\sin\alpha}{W\cos\alpha}$$
$$= \tan\alpha.$$

Now the fraction F divided by R is equal to μ, the coefficient of friction. Hence

$$\mu = \tan\alpha.$$

The angle α can be determined by actual measurement, and from a table of natural tangents we can find its tangent. Hence the value of μ can be found.

For most pairs of surfaces μ is less than unity.

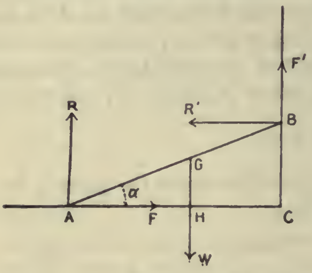
The angle α, the steepest inclination of the plane AB at which the body FEG will rest upon AB, is called the **angle of friction** or the **angle of repose.**

85. Examples of Problems involving Friction.

Example 1.—A uniform ladder rests on a floor against a wall. If the coefficient of friction between the ladder and both floor and wall is $\frac{1}{3}$, what is the least inclination to the horizon at which the latter will rest?

The ladder is, of course, supposed to be in a plane perpendicular to the wall. Let the figure represent the position of the ladder AB when just on the point of sliding down. There are in all five forces acting on the ladder:

(1) The weight W, which may be supposed to act at G, the middle point of AB.



(2) The reaction R of the ground at the end A of the ladder which rests on the floor, acting perpendicular to the floor.

(3) The friction F along the floor AC, tending to prevent the end A from sliding.

(4) The reaction R′ of the wall at the end B of the ladder which rests on the wall, acting perpendicular to the wall and therefore horizontally.

(5) The friction F′ acting upwards at B, tending to prevent the end B from sliding down.

These five forces are in equilibrium, and the conditions of Article 64 must be satisfied, that is, the sums of the vertical and horizontal components must be zero, and the algebraical sum of the moments of the five forces round any point in the plane must be zero.

Resolving vertically, we have

$$R + F' = W.$$

Resolving horizontally, we have

$$F = R'.$$

Since the ladder is just on the point of sliding down, the maximum amount of friction is called into account, and therefore the friction at A or B is found by multiplying the pressure by the coefficient of friction. Hence

$$F = \tfrac{1}{3}R, \text{ and } F' = \tfrac{1}{3}R'.$$

Substituting in the above equations, we get

$$R + \tfrac{1}{3}R' = W$$
$$\tfrac{1}{3}R = R'.$$

Solving these equations for R and R′, we find

$$R = \tfrac{9}{10}W, \ R' = \tfrac{3}{10}W.$$

Lastly, by equating the algebraical sum of the moments round A to zero, we get

$$R'.BC + F'.AC = W.AH,$$

or $$R'.BC + \tfrac{1}{3}R'.AC = W.AH, \quad (\text{since } F' = \tfrac{1}{3}R')$$

or $$\tfrac{3}{10}W.BC + \tfrac{1}{3}.\tfrac{3}{10}W.AC = W.AH, \quad (\text{since } R' = \tfrac{3}{10}W)$$

or $$3\,BC + AC = 10\,AH, \quad \left(\text{dividing by } \frac{W}{10}\right)$$

or $$3\,AB\sin\alpha + AB\cos\alpha = 10.\tfrac{1}{2}\,AB\cos\alpha,$$

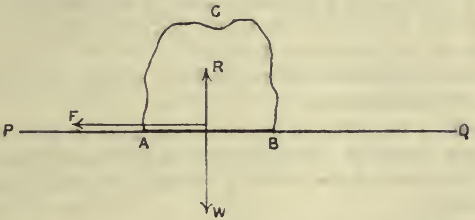
or $$3\sin\alpha = 4\cos\alpha, \quad (\text{dividing by } AB)$$

from which $$\tan\alpha = \tfrac{4}{3} = \tan 53^\circ\, 8'.$$

Hence the least inclination to the horizon at which the ladder will rest is 53° 8′.

Example 2.—A body is projected along a horizontal plane with a velocity of 20 feet per second. If the coefficient of kinetic friction between the body and the plane is $\frac{2}{5}$, after what time will the body be brought to rest, and how far will it travel in that time?

Let the body ABC be projected from P in the direction PQ, then at any time during the motion the forces acting on the body are



(1) the weight W vertically downwards, (2) the reaction R of the plane vertically upwards, and (3) the friction F in the direction QP, tending to retard the motion of the body.

If m be the mass of the body, the weight of the body in absolute units is mg. Now the weight is evidently equal to the reaction R; hence the reaction in absolute units is mg, and the friction, R multiplied by the coefficient of friction is $\frac{2}{5}mg$. This is the only force tending to retard the motion of the body, and since the mass is m, the retardation is

$$\frac{\frac{2}{5}mg}{m} \text{ or } \tfrac{2}{5}g.$$

We have thus a case of motion in which a body starts with a velocity of 20 feet per second, and has a negative acceleration of $\frac{2}{5}g$ per second. If t denote the time, s the space required, we have by the second special case of Article 27,

$$20^2 - 2 . \tfrac{2}{5} g . s = 0,$$
$$20 - \tfrac{2}{5} gt = 0,$$

from which (taking $g = 32{\cdot}2$),

$$s = \frac{500}{32{\cdot}2} = 15{\cdot}53 \text{ feet very nearly.}$$
$$t = \frac{50}{32{\cdot}2} = 1{\cdot}553 \text{ seconds very nearly.}$$

EXAMPLES XXII.

1. A ladder rests inclined at an angle of 30° to the vertical, with one end on a floor and the other against a vertical wall. A man ascends the ladder, and the ladder is on the point of slipping when his C.G. is half-way up the ladder.

(1) If the wall is perfectly smooth, so that there is no friction between the wall and the ladder, find the coefficient of friction between the ladder and the floor.

(2) If the floor and the wall are equally rough, find the coefficient of friction.

2. A train is travelling at the rate of 45 miles an hour when the steam is shut off. The train is brought to rest by the friction of the wheels on the rails in 3 minutes. Find the coefficient of friction.

3. A body is projected along a rough horizontal plane with a velocity of 30 metres per second, the coefficient of friction between the body and the plane being $\frac{1}{4}$. After what time will the body be brought to rest, and how far will it travel in that time? Take $g = 981$.

4. Show that the acceleration of a body down a plane inclined to the horizon at an angle α is

$$g \sin \alpha - \mu g \cos \alpha,$$

where g represents the acceleration of gravity and μ the coefficient of friction between the body and the plane.

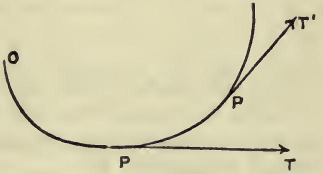
5. A log of wood is sliding down a timber slide, the coefficient of friction being ${\cdot}45$, and the slide being inclined to the horizon at an angle of 30°. How long will the log, starting from rest, take to slide down 12 yards, and what will be its velocity at the end of that time? Take $g=32$.

CHAPTER VII.

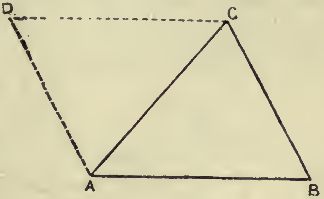
HODOGRAPH. UNIFORM CIRCULAR MOTION.

86. Change of Velocity.

In Chapter II. we considered what is meant by the change of velocity of a body moving in a straight line. We are now about to consider what is meant by the change of velocity of a body moving in a curve.



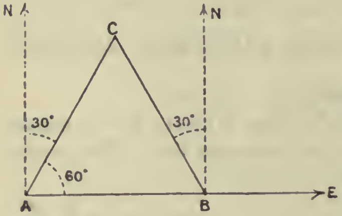
A body is moving in any curve OPP′. The directions of motion at the points P and P′ are along the tangents PT and P′T′ respectively. Let two lines AB and AC be drawn through any point A representing in magnitude and direction the velocities at the points P and P′ respectively; then the change of velocity between the two points P and P′ is represented in magnitude and direction by BC.



Complete the parallelogram BD; then, AD being equal and parallel to BC, the change of velocity may be represented by AD. Now AD is the velocity which must be compounded with the velocity AB to produce the velocity AC. Hence we say that the change of velocity between the two points P and P′ is the velocity which must be compounded with the velocity at P to produce the velocity at P′.

EXAMPLE.—A body at a certain time is moving with a velocity of 4 feet per second eastwards, and at a subsequent time is moving with a velocity of 4 feet per second N. 30° E. What is the change of velocity and magnitude and direction?

Let AB represent the velocity eastwards and AC the velocity N. 30° E.; then the change of velocity is represented by BC. Now it is evident that the angle BAC = 60°, and AB = AC, so that ABC is an equilateral triangle. Hence BC = AB, and the angle CBA = 60°, and therefore the angle CBN = 30°.



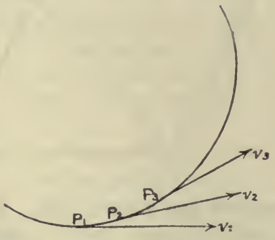
The change of velocity therefore is 4 feet per second per second in a direction N. 30° W.

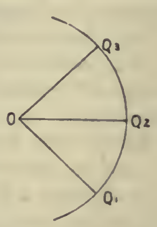
87. Extended definition of Acceleration.

Taking the extended meaning of change of velocity explained in the preceding Article, we define acceleration as the rate of change of velocity.

88. Hodograph.

If a body is moving in any manner in a straight line or curve, and if straight lines be drawn from any point representing in magnitude and direction the velocity at every point of the curve, the extremities of these lines will form a curve which is called the *hodograph* of the curve in which the body is moving.





Let a body be moving in the curve $P_1 P_2 P_3 \ldots$, and let the velocities at the points P_1, P_2, P_3, ... be V_1, V_2, V_3, ...

Let straight lines be drawn from any point O representing in magnitude and direction the velocities V_1, V_2, V_3, ... Let Q_1, Q_2, Q_3, ... be the extremities of these lines, then it is evident that if the points P_1, P_2, P_3, ... are taken very close together on the curve $P_1P_2P_3$ .... the lines OQ_1, OQ_2, OQ_3, ... will be very close together, and the points Q_1, Q_2, Q_3, ... will be points on a curve $Q_1Q_2Q_3$ .... which is the *hodograph* of the curve $P_1P_2P_3$ ....

To every point in the curve there will correspond a point on the hodograph, and we may therefore imagine that as the body describes the curve another body describes the hodograph.

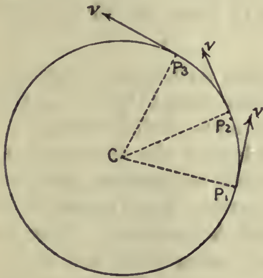
The hodograph possesses the two following properties, which we give without proof:

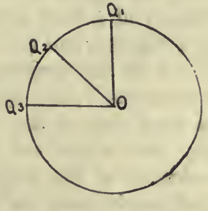
(1) *The velocity at any point of the hodograph of the imaginary body describing the hodograph is equal to the acceleration of the body describing the curve at the corresponding point of the curve.*

(2) *The direction of the velocity at any point of the hodograph of the imaginary body describing the hodograph is the same as the direction of acceleration of the body describing the curve at the corresponding point of the curve.*

89. Uniform Motion in a Circle.

Let $P_1P_2P_3$ .... be a circle described with uniform *velocity V; then if from any point lines OQ_1, OQ_2, OQ_3, ... be drawn representing in magnitude and direction the velocities at the points P_1, P_2, P_3, ..., it is evident that these lines will all be equal. Hence the points Q_1, Q_2, Q_3, ... will all lie on the





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* That is. uniform in magnitude; more properly uniform *speed*.

circumference of a circle. Therefore the hodograph of a circle described with uniform velocity is a circle.

Again, since the circle $P_1P_2P_3 \ldots$ is described with uniform velocity, the arcs P_1P_2, P_2P_3, ... described in equal times will be equal, and therefore the angles subtended at the centre P_1CP_2, P_2CP_3, ... will be equal. But the angle P_1CP_2 is equal to the angle between the tangents at P_1 and P_2, the angle P_2CP_3 is equal to the angle between the tangents at P_2 and P_3, and so on. Hence since the lines OQ_1, OQ_2, OQ_3, ... are parallel to the tangents at P_1, P_2, P_3, ... respectively, it follows that the angles Q_1,OQ_2, Q_2OQ_3, ... described in equal times are equal. Hence also the arcs Q_1Q_2, Q_2Q_3, ... described in equal times are equal, or what is the same thing, the hodograph is described with uniform velocity. It is evident also that one revolution of the hodograph is made in the same time as one revolution of the curve.

Lastly, since the hodograph is a circle the direction of the velocity in the hodograph at any point Q_1 is along the tangent to the circle at Q_1, that is, is perpendicular to the radius OQ_1. Now the radius OQ_1 is parallel to the tangent at P_1 or perpendicular to the radius CP_1. Hence the direction of velocity at any point of the hodograph is the same as the direction of the radius at the corresponding point of the curve.

90. Acceleration of a Body moving in a Circle with Uniform Velocity.

Taking the figures of the preceding Article, let V represent the velocity of a body describing the circle $P_1P_2P_3 \ldots$ with uniform velocity, and let R be the radius of that circle.

We saw in the preceding Article that the hodograph is a circle of radius V, and that this circle is described uniformly in the same time as the original circle. Let v represent the velocity with which the hodograph is described. Then the circumference of the original circle being $2\pi R$, and that of the hodograph being $2\pi V$, we see that the distance $2\pi R$ is described with uniform velocity V in the same time as the distance $2\pi V$ with the velocity v. Hence by Article 13

$$\frac{v}{2\pi V}=\frac{V}{2\pi R},$$

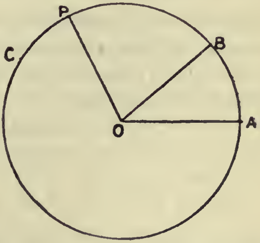
from which $$v=\frac{V^2}{R}.$$

Now the velocity in the hodograph is equal to the acceleration in the curve, and the direction of the velocity in the hodograph is the same as the direction of acceleration in the curve. Collecting our results, we say therefore that a point moving in a circle with uniform velocity V has continually an acceleration measured by $\frac{V^2}{R}$ always directed towards the centre of the circle.

It may appear to the student at first sight anomalous that a body can have an acceleration while it is moving with uniform velocity, but he must remember that the word acceleration is used in its extended meaning, viz., rate of change of velocity whether that change takes place in the direction of motion or not. When a body moves in a circle with uniform velocity there is no change in the magnitude of the velocity, but there is a continual change in the direction, and the rate of that change, which is uniform, is measured by $\frac{V^2}{R}$.

91. Angular Velocity of a Body Describing a Circle with Uniform Velocity.

If a body is describing a circle, the angle through which the radius joining the body with the centre revolves in one unit of time is called the **angular velocity** of the body round the centre.



If the velocity of the body is uniform the angular velocity will be uniform, and will be measured by the angle in radians through which the radius revolves in one unit of time. Taking a second as the unit of time, let AB be the arc described by the body in one second, then the number of radians in the angle AOB is the angular velocity of the body. Let V denote the linear, ω the angular velocity, and R the radius; then by Trigonometry

$$\text{arc AB} = \text{angle AOB} \times \text{radius},$$

and therefore

$$V = \omega R \text{..............................(1)}$$

Also if the arc AP is described in t seconds, the arc AP will be t times the arc AB, and the angle AOP will be t times the angle AOB, that is, the angle described by the revolving radius is

$$\omega t.$$

If T denote the time in seconds taken by the body to describe the circle, then the radius revolves through the angle 2π in T seconds, and therefore through the angle $\frac{2\pi}{T}$ in one second. Hence

$$\omega = \frac{2\pi}{T} \text{..............................(2)}$$

From formulæ (1) and (2) we have

$$\omega = \frac{V}{R} = \frac{2\pi}{T}.$$

Example.—A point moves in the circumference of a circle of r feet radius with an angular velocity of ω radians per second. What time does it take to go over s feet of its path?

By formula (1) the velocity is ωr feet per second, and it will therefore take $\frac{s}{\omega r}$ seconds to move over s feet.

92. We can express the acceleration of a body describing a circle uniformly in terms of the angular velocity, ω, and also in terms of T, the time of describing the circle.

Since $V = \omega R$, the acceleration is

$$\frac{V^2}{R} = \frac{\omega^2 R^2}{R} = \omega^2 R \text{..........................(1)}$$

Also, if T be the time of describing the circle, called the periodic time, $\omega = \frac{2\pi}{T}$, and therefore the acceleration in terms of T is

$$\left(\frac{2\pi}{T}\right)^2 R \text{..............................(2)}$$

93. Centrifugal Force.

We have seen that a body describing a circle uniformly has continually an acceleration towards the centre. It follows from Newton's second law of motion that a body moving in a circle must be continually acted on by a force towards the centre of the circle. The magnitude of this force in absolute units will be found by multiplying the mass of the body by the acceleration. The acceleration can be expressed in the three forms

$$\frac{V^2}{R}, \quad \omega^2 R, \quad \left(\frac{2\pi}{T}\right)^2 R,$$

and, if m denote the mass of the body, the force can in like manner be expressed in the three forms

$$\frac{mV^2}{R}, \quad m\omega^2 R, \quad m\left(\frac{2\pi}{T}\right)^2 R.$$

We have thus arrived at the result that in order to keep a body moving in a circle there must be continually acting on the body a force directed towards the centre of the circle, and we have given three equivalent formulæ for the measure of this force in absolute units. This force has no effect on the actual velocity of the body, its only effect being to deflect the body from the tangent along which it is moving at any particular instant. If at any point this force were to cease the body would continue to move along the tangent to the circle at that point.

Before the time of Newton it was known that force was necessary to keep a body moving in a circle, and it was supposed that this force was required to balance another force tending to drive the body away from the centre of the circle. Hence this supposed force was called **Centrifugal Force.** We now know that this supposed force is nothing more than the tendency of the body in virtue of its inertia to keep moving in a straight line.

Although the idea of centrifugal force is now known to be false, the term is still used to denote this tendency of a body to preserve its state of motion in a straight line. The three expressions given in this Article for the force necessary to overcome this tendency in the case of a body moving uniformly in a circle are often referred to as expressions for the centrifugal force.

Many examples of centrifugal force are met with in everyday life. Thus there is centrifugal force when a railway train is rounding a curve, this force being balanced by the pressure of the outer rail on the wheels of the engine and carriages. It is on account of centrifugal force that the balls in Watt's governor for the steam-engine open out when revolving rapidly. The figure of the earth, an oblate spheroid—that is, a sphere flattened at the poles—is due to the action of centrifugal force when the earth was cooling down ages ago from the liquid to the solid state.

Example 1.—A toy car whose mass is ½ lb. runs at the rate of 5 miles an hour on a level circular railway of 3 feet radius. Calculate the horizontal pressure on the rails.

The horizontal pressure on the rails is equal to the centrifugal force, and the formula for that force is

$$\frac{mV^2}{R}.$$

Adopting British units,

$$m=\tfrac{1}{2},\ \ R=3,\ \ V=\frac{5\times 1760\times 3}{60\times 60}=\frac{22}{3},$$

and therefore the horizontal pressure on the rails is

$$\frac{1}{2}\cdot\frac{\left(\frac{22}{3}\right)^2}{3}=8{\cdot}963 \text{ poundals.}$$

Taking $g=32$, this is equivalent to ·28 of the weight of one pound.

Example 2.—A stone whose mass is 2 lbs. is swung round in a horizontal circle 3 times a second at the end of a string 3 feet long. What is the tension of the string?

The tension of the string is equal to the centrifugal force. Since we know the periodic time T, we shall use the formula

$$m\left(\frac{2\pi}{T}\right)^2 R$$

for the centrifugal force.

Adopting British units,

$$m=2,\ R=3,\ T=\tfrac{1}{3},$$

and therefore the tension of the string is

$$2 \times \frac{4\pi^2}{(\frac{1}{3})^2} \times 3$$

$$=2160 \text{ poundals (taking } \pi^2=10)$$
$$=67\tfrac{1}{2} \text{ pounds' weight (taking } g=32).$$

EXAMPLES XXIII.

1. A body is moving in a circle with an angular velocity of 3 radians per second. If the radius of the circle is 10 feet, what is the linear velocity of the body?
2. A body moving in a circle with a linear velocity of 20 feet per second passes over a quadrant of the circle in 3 seconds. What is the angular velocity, and what is the radius?
3. A locomotive whose mass is 19 tons passes round a curve in the shape of part of a circle whose radius is half a mile with a velocity of 35 miles per hour. What is the centrifugal force?
4. A stone is swung round in a horizontal circle, first at the end of one string and then at the end of another twice as long, so as to make the same number of revolutions in a minute in both cases. Compare the tensions of the two strings.
5. What force would be required to keep a body whose mass is 10 lbs. rotating in a circle of 15 feet radius, (1) with a velocity of 30 feet per second, (2) if it were making one complete revolution in a second?
6. A wire is found to break with a weight of 100 lbs. If a mass of 10 lbs. is fastened to a piece of the same wire and swung round in a horizontal circle 20 feet radius, what velocity will the 10 lbs. have when the wire breaks?
7. A mass of 2 lbs. is whirled round on a smooth horizontal plane by a string 4 feet long attached to a point in the plane. If the string can support 9 lbs. without breaking, what is the greatest velocity which can be given to the mass so as not to break the string?
8. Two bodies A and B describe circles with constant velocities, the radius of A's circle being 390 times that of B's circle. A moves round its circle once while B moves round its circle 13 times. If A's mass is 91 times B's mass, compare the force acting on A with the force acting on B.

9. Taking the equatorial radius of the earth to be 4000 miles, show that the centrifugal force of a given mass at the equator, due to the rotation of the earth round its axis once in every 24 hours, is equal to about $\frac{1}{289}$ of the weight of the given mass.

10. Show that everything else being the same the centrifugal force varies as the square of the angular velocity, or inversely as the square of the time of one complete revolution.

11. Hence find how much faster the earth would require to revolve in order that the value of g at the equator would be nil.

12. What is the force in grammes' weight required to keep a mass of 20 grammes revolving 30 times in a second in a circle 4 centimetres diameter? (Take $g=981$.)

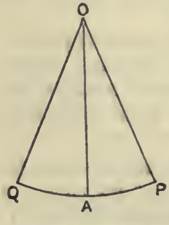
Chapter VIII.

SIMPLE PENDULUM. MOMENT OF INERTIA. COMPOUND PENDULUM.

94. Simple Pendulum.

A simple pendulum is defined to be a mass of infinitesimal magnitude hung at the end of an inextensible weightless string.

A simple pendulum is therefore an ideal conception. The finest string has a certain weight, and every mass, however small, has a certain magnitude. If, however, we take a very fine thread and attach to the end of it a very small body, such as a ball of lead, we have a close approximation to the ideal simple pendulum. We shall suppose that our simple pendulum consists of such an arrangement as this. If such a pendulum be suspended from a point, it will, of course, hang in equilibrium with the string vertical, and if displaced will oscillate backwards and forwards through this position, finally coming to rest in this position.



If OA be the position of equilibrium of a simple pendulum oscillating through the equal arcs AP and AQ on either side of the vertical, then the angle AOP or the equal angle AOQ is called the **amplitude** of vibration, and the time the pendulum takes to swing from P to Q and back again to P is called the time of a complete vibration. The time from P to Q, or from Q to P, will of course be equal to half the time of a complete vibration.

95. Time of Vibration of a Simple Pendulum.

The simple pendulum possesses this remarkable property, that, as long as the amplitude is less than 2° or 3°, the time of vibration is independent of the amplitude, and depends only on its length and on the value of the acceleration of gravity. Thus a simple pendulum will vibrate in the same time at the same place whether the amplitude be 2° or 1° or only 10″.

The actual formula, which we give without proof, for the time of vibration* of a simple pendulum oscillating through a small arc on either side of the vertical, is

$$t = \pi\sqrt{\frac{l}{g}},$$

where t represents the time in seconds of *half* a complete vibration, π the ratio of the circumference of a circle to its diameter 3·1416, l the length of the pendulum, and g the acceleration of gravity at the place.

If British units are used, l must be expressed in feet and g in feet and seconds. If C.G.S. units are used l must be expressed in centimetres and g in centimetres and seconds.

Example 1.—Find the time of vibration of a simple pendulum 4 feet long at a place where $g = 32{\cdot}2$. Using the formula we find, putting $l = 4$, $g = 32{\cdot}2$, the time to be

$$3{\cdot}1416\sqrt{\frac{4}{32{\cdot}2}} = 1{\cdot}107 \text{ seconds nearly.}$$

Example 2.—How many vibrations will a simple pendulum 99·8 centimetres long make in 24 hours at Edinburgh, the acceleration of gravity at that place in C.G.S. units being 981·54?

---

* Hereafter when we speak of the time of vibration of a pendulum we shall always mean the time of half a complete vibration.

The time of one vibration is

$$3{\cdot}1416\sqrt{\frac{99{\cdot}8}{981{\cdot}54}}=1{\cdot}0017 \text{ seconds.}$$

Since there are $24\times60\times60$ seconds in a day, the number of vibrations made in one day is

$$\frac{24\times60\times60}{1{\cdot}0017}=86253.$$

Since there are 86400 seconds in a day, a clock regulated by such a pendulum would lose 86400 − 86253, or 147 seconds in one day.

96. The Seconds' Pendulum.

The seconds' pendulum at a given place is a pendulum of such a length as to beat seconds at that place. To find the length of the seconds' pendulum we put $t=1$ in the formula of the preceding Article. Thus, l denoting the length of the seconds' pendulum,

$$1=\pi\sqrt{\frac{l}{g}},$$

from which $$l=\frac{g}{\pi^2}.$$

Example 1.—What is the length of the seconds' pendulum at Greenwich, where the acceleration of gravity in British units is 32·1912?

The length is

$$\frac{32{\cdot}1912}{(3{\cdot}1416)^2} \text{ feet}=39{\cdot}139 \text{ inches.}$$

Example 2.—A seconds' pendulum was lengthened, and the increase in the time of vibration was $\frac{1}{80}$ of a second. By how much was the pendulum lengthened?

Let l denote the original length, and l' the increased length; then, since the time of vibration with the original length was 1 second, the time of vibration with the increased length was $\frac{81}{80}$ seconds. Hence

$$1=\pi\sqrt{\frac{l}{g}}, \quad \frac{81}{80}=\pi\sqrt{\frac{l'}{g}},$$

from which $$l=\frac{g}{\pi^2}, \quad l'=\frac{g}{\pi^2}\left(\frac{81}{80}\right)^2.$$

By division $$\frac{l'}{l}=\left(\frac{81}{80}\right)^2,$$

and therefore $$\frac{l'-l}{l}=\frac{81^2-80^2}{80^2}$$

$$=\frac{161}{6400}.$$

Thus $$l'-l=\frac{161}{6400}\,l,$$

or the increase in length is $\frac{161}{6400}$ of the original length.

97. Determination of the Acceleration of Gravity by Pendulum Experiments.

From the formula of Article 95 we obtain, on solving for g,

$$g=\frac{\pi^2 l}{t^2},$$

a formula which gives the value of the acceleration of gravity at any place in terms of l, the length of a simple pendulum, and t the time of vibration of the pendulum at that place. Experiments have been made with the pendulum at numerous places on the earth's surface, and the values of g have been deduced from these experiments by means of the above formula. Since, however, any actual pendulum is only an approximation to the ideal simple pendulum, the values of g determined by such experiments are not strictly accurate.

The general result shown by pendulum experiments is that the value of g is least at the equator and increases as we proceed from the equator to either pole. This variation in the value of g is due to two causes: (1) Since the earth is an oblate spheroid a body is farther from the centre of the earth at the equator than at any other latitude, and is nearer the centre the higher the latitude. But the earth attracts bodies with a force which varies inversely as the square of the distance from the centre, and therefore the force of the earth's attraction on any given mass is greater the higher the latitude. (2) The centrifugal force, due to the earth's diurnal revolution on its axis, is greater at the equator than at any other latitude. We saw that a mass m moving in a circle, radius R, with an angular velocity ω had a centrifugal force $m\omega^2R$. Now the angular velocity due to the earth's revolution is the same for all latitudes, but the radius of the circle in which a body revolves is less the higher the latitude. The radius of this circle at the equator is 4000 miles, at 60° latitude it is 2000 miles, and generally is less the nearer we approach the pole, where it becomes nil. Now the centrifugal force due to the earth's rotation diminishes the weight of a body, and therefore for this cause also the weight of a body and the acceleration of gravity are less the nearer the equator.

EXAMPLES XXIV.

[In working this set of examples the student will find it useful to employ a Table of Logarithms. Take $\pi = 3{\cdot}1416$.]

1. How many vibrations would a pendulum 10 feet long make in 15 minutes at a place where $g = 32{\cdot}2$?

2. Find the lengths in inches of the seconds' pendulum at the following places, given in each case the acceleration of gravity in British units:—

Place.	Value of g.
(1) Paris,	32·182
(2) Glasgow,	32·2
(3) New York,	32·159
(4) Equator,	32·088

3. Find the acceleration of gravity in C.G.S. units at each of the following places, given in each case the length of the seconds' pendulum in centimetres:—

Place.	Length of seconds' pendulum.
(1) Equator,	99·1
(2) Latitude 60°	99·5
(3) Pole,	99·6

4. A pendulum 39·2 inches long vibrates seconds at a certain place. Find the acceleration of gravity at that place.

   Also find how much faster the pendulum would beat at a place where the force of gravity is ¼ per cent greater.

5. A pendulum 37·8 inches long is found to make 183 beats in 3 minutes at a certain place. Find the acceleration of gravity at that place.

6. A seconds' pendulum is lengthened 1 per cent. How much would it lose in a day?

7. At a certain place the length of the seconds' pendulum is 39·125 inches. Find the length of the pendulum making 100 vibrations in a minute.

8. If a simple pendulum which should beat seconds lose 20 seconds in a day, find by how much its length should be altered.

9. If a clock gains half an hour per week, by what percentage of its own length must the pendulum be altered to make the clock keep correct time?

10. If a clock were regulated so as to keep correct time at a place where g is 32·24, what would be its daily error at a place where $g=32{\cdot}09$, and what proportionate change in the length of the pendulum would be needed to make it keep exact time at the latter place?

98. Moment of Inertia.

If, as in Article 67, we conceive a body made up of an infinite number of material particles, and if the mass of each particle be multiplied by the square of its distance from a given line, the sum of all the products so formed is called the **moment of inertia** of the body with respect to the given axis.

In general, in order to find moments of inertia it is necessary to use the integral calculus. In cases in which the body consists of a finite number of particles whose masses and positions are known, the moment of inertia of the body round any axis can be found by elementary algebra. Thus if m denote the mass of each of four equal masses, considered as particles, placed one at each corner of a square whose side is of length $2a$, the moment of inertia of the four masses round an axis through the centre of the square perpendicular to the plane of the square, is

$$m \,.\, 2a^2 + m \,.\, 2a^2 + m \,.\, 2a^2 + m \,.\, 2a^2 \text{ or } 8ma^2.$$

In like manner the moment of inertia of three equal particles each of mass m, placed one at each angular point of an equilateral triangle, with respect to an axis through the C.G. of the triangle perpendicular to the plane of the triangle, is

$$m \cdot \frac{4a^2}{3} + m \cdot \frac{4a^2}{3} + m \cdot \frac{4a^2}{3} \text{ or } 4ma^2,$$

$2a$ being the length of a side of the triangle.

99. Radius of Gyration.

If M denote the mass of a body and k denote a length such that Mk^2 is the moment of inertia of the body with respect to a given axis, then k is called the **radius of gyration** of the body with respect to that axis.

In the first example given in the preceding Article we saw that the moment of inertia was $8ma^2$ or $4m \times 2a^2$, and therefore, since the mass of the body is $4m$, the square of the radius of gyration with respect to the given axis is $2a^2$. Similarly, it would be found that the square of the radius of gyration in the second example is $\frac{4a^2}{3}$.

100. Table of Moments of Inertia.

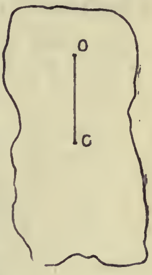
Subjoined we give the moments of inertia and the values of k^2, the square of the radius of gyration, for a few bodies of simple shape with respect to specified axes. The bodies are supposed to be of uniform density throughout, and the mass of each body is represented by M.

	Moment of inertia.	Value of k^2.
Rod, length $2a$, about an axis through its middle point perpendicular to its length, ...	$\frac{1}{3} a^2 M$	$\frac{1}{3} a^2$
Rod, length $2a$, about an axis through one end perpendicular to its length,	$\frac{4}{3} a^2 M$	$\frac{4}{3} a^2$
Solid cylinder, radius a, about its axis,	$\frac{a^2}{2} M$	$\frac{a^2}{2}$
Rectangular block, length of edge a, about an axis through middle point of one face perpendicular to that face, ...	$\frac{2}{3} a^2 M$	$\frac{2}{3} a^2$
Solid sphere, radius a, about a diameter, ...	$\frac{2}{5} a^2 M$	$\frac{2}{5} a^2$

101. Compound Pendulum.

We saw how the acceleration of gravity at a place could be found approximately by means of experiments performed on Attwood's machine. We have also seen that an approximate determination could be made by considering a small mass hung at the end of a thin string as a simple pendulum, and finding the time of vibration. The most accurate method is by observations made on a **compound pendulum.** A compound pendulum is a body supported on a horizontal axis, and free to swing about that axis under the action of the force of gravity. Thus any pendulum in which the masses of the string or rod and of the bob are taken into account is a compound pendulum.

Let the figure represent a vertical section through the centre of gravity, G, of a compound pendulum which is free to oscillate about a horizontal axis through the point O. The pendulum will hang in equilibrium with the point G vertically under the point O, and if displaced from that position will oscillate backwards and forwards through that position. If the line OG does not oscillate through more than two or three degrees on either side of the vertical position, the time of oscillation will, as in the case of the simple pendulum, be independent of the amplitude, and will depend only on the position of G with reference to O and the moment of inertia of the body about the axis through O.



If h denote the distance OG and Mk^2 the moment of inertia of the body with respect to the axis through O, M being the mass of the body, then the time of a small oscillation, that is, of half a complete vibration, is

$$\frac{\pi k}{\sqrt{gh}}.$$

Let l be the length of a simple pendulum which would make a small vibration in the same time: then, by Article 95,

$$\pi\sqrt{\frac{l}{g}}=\frac{\pi k}{\sqrt{gh}},$$

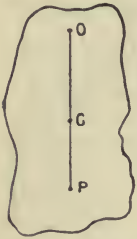
from which $$l=\frac{k^2}{h}.$$

The length l is said to be the length of the **equivalent simple pendulum.**

102. Centres of Oscillation and Suspension.

In the figure of the preceding Article the point O is called the **centre of suspension.** If OG be produced to P so OP is equal to $\frac{k^2}{h}$, then the point P is called the **centre of oscillation,** corresponding to the centre of suspension O.

The relation between the centre of suspension and the centre of oscillation is reciprocal. If the body be set to oscillate about a horizontal axis through P, and if k' be the radius of gyration round this axis and h' the distance GP, then the distance PO is $\frac{k'^2}{h'}$, and the time of oscillation about this axis through P is equal to the time of oscillation about the parallel axis through O. From this we infer the following important proposition:—



If on opposite sides of the C.G. of a body of irregular form there can be found two axes about which the times of oscillation of the body are the same, the distance between those two axes is equal to the length of the equivalent simple pendulum.

The student will easily see why we say "of irregular form:" the times of oscillation about any pair of axes at equal distances from the C.G. will be the same if the body is perfectly symmetrical round its C.G.

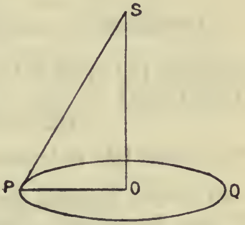
Thus by observing the time of vibration of a compound pendulum about a certain axis, and by finding by trial another axis on the opposite side of the centre of gravity about which the time of oscillation is the same, it is only necessary to measure the distance between the two axes to find with accuracy the length of the ideal simple pendulum vibrating in the same time. We can then make use of the formula of Article 97 for determining the acceleration of gravity at the place where the experiment was made.

This method was used by Captain Kater for determining the acceleration of gravity at different stations in Great Britain.

103. Conical Simple Pendulum.

A simple pendulum in which the mass at the end of the string describes a horizontal circle is called a **conical simple pendulum.**

The figure represents a conical pendulum. The pendulum SP is suspended from the point S, and the mass P describes the horizontal circle PQ about the vertical axis SO.



Let l be the length of the pendulum SP, and T the time of one complete revolution of the pendulum. Then we can express the radius OP of the horizontal circle, the height OS of the point of suspension above this circle, and the cosine of the angle PSO which the pendulum makes with the vertical axis, all in terms of l and T.

The only external forces acting on the mass P are the tension of the string along PS and the weight of the mass P vertically downwards. The resultant of these two forces must therefore balance the centrifugal force due to the motion of P in the horizontal circle PQ. Hence we may look upon these three forces—(1) the tension of the string along PS, (2) the weight of the mass P vertically downwards, and (3) the centrifugal force along OP—as forming a system of three forces in equilibrium. Since these three forces act parallel to the sides PS, SO, OP respectively of the triangle PSO, it follows from the Triangle of Force that they must be proportional to the lines SP, SO, OP respectively.

If m be the mass of the particle P, the weight vertically downwards is mg, and the centrifugal force is $m\left(\frac{2\pi}{T}\right)^2 . OP$; hence

$$SO : OP = mg : m\left(\frac{2\pi}{T}\right)^2 . OP,$$

or $$\frac{SO}{OP} = \frac{gT^2}{4\pi^2 OP},$$

from which $$SO = \frac{gT^2}{4\pi^2} \quad \text{.......... (1)}$$

From this formula we infer that at a given place SO depends only on T, the time of one complete revolution.

Again $$SO = SP \cos PSO;$$

or $$\frac{gT^2}{4\pi^2} = l \cos PSO;$$

from which $$\cos PSO = \frac{gT^2}{4\pi^2 l} \quad \text{.......... (2)}$$

Equation (1) gives SO in terms of T; and therefore, since (Euclid I. 47)

$$OP^2 = SP^2 - SO^2 = l^2 - SO^2, \quad \text{.......... (3)}$$

we can find OP^2 in terms of l and T.

Example.—A conical pendulum is spinning round a vertical axis 20 times in a second. Find the distance in inches of the mass at the end of the pendulum below the horizontal plane through the point of suspension.

Taking British units, $T = \frac{1}{20}$, $g = 32$, and therefore the height of the pendulum is

$$\frac{gT^2}{4\pi^2} = \frac{32 \times (\frac{1}{20})^2}{4 \times 9{\cdot}87} \text{ feet}$$

$$= \frac{1}{41} \text{ inch very nearly.}$$

EXAMPLES XXV.

1. An axis is drawn through the centre of gravity of a body whose mass is M; a second axis is drawn parallel to the former, and at a distance h from it. If I denote the moment of inertia of the body with respect to the first axis, then the moment of inertia with respect to the second axis is $I + Mh^2$.

Assuming this proposition, find, with the help of the table of moments of inertia in Article 100, the following moments of inertia:—

(1) The moment of inertia of a circle about an axis through any point of the circumference perpendicular to the plane of the circle: radius $= a$.

(2) The moment of inertia of a square about an axis through an angular point perpendicular to the plane of the square: edge $= a$.

(3) The moment of inertia of a fine wire bent into the form of a circle about an axis through any point of the wire perpenpendicular to its plane: radius $= a$.

(4) The moment of inertia of a uniform rod coinciding with one edge of a cube about another edge which neither meets nor is parallel to the former edge: edge $= a$.

2. A rod vibrating about an axis through one end makes 66 oscillations in one minute. Find another point in it, from which if it be suspended it will still make 66 oscillations per minute.

3. A simple conical pendulum 3 feet long is swinging round a vertical axis 3 times in a second. Find the height of the point of suspension above the circle which the end of the pendulum is describing, and the cosine of the angle which the pendulum is making with the vertical.

4. An equestrian standing on a horse gallops in a ring 24 feet in diameter at the rate of 10 miles an hour. At what angle must the horse and the rider be inclined to the vertical?

5. A man on a 50-inch bicycle in going round a circle of 50 feet radius leans inwards so that the lowest point of the wheel is 3 inches farther out than the highest. Find his speed in feet per second.

CHAPTER IX.—WORK AND ENERGY.

104. Work.

A force acting on a body is said in dynamics to do **work** on the body when the body moves in a direction inclined at any angle less than a right angle to the direction in which the force acts. Thus the force of gravity does work on a falling body, the tension of the hawser by which a tug-boat tows a vessel does work on the vessel, the pressure of steam in expanding in the cylinder of a steam-engine does work on the piston, &c.

If the body moves in the direction in which the force acts, the work done by the force, while the body is passing over a given space, is measured by the product of the force and the space through which the body moves. Thus the force required to lift a mass vertically is equal to the weight of the mass, and therefore if the mass is lifted through a given height, the work done by the lifting force is the product of that force and the height.

If the body moves in a direction inclined to the direction in which the force acts, the work done is found by multiplying the component of the force in the direction in which the body moves by the distance through which the body moves. Thus if T be the tension of the rope by which a horse on the towing-path tows a canal boat, and α the angle the direction of the rope makes with the direction in which the boat is moving, the component of the tension in the direction of the boat's motion is $T\cos\alpha$, and the work in towing the boat over a given distance will be measured by the product of $T\cos\alpha$ and the distance.

It follows from the definition that there are two cases in which forces do no work: (1) Forces in equilibrium do no work. (2) A force acting on a body does no work when the body moves in a direction at right angles to the direction in which the force acts. As an example, take the case of the simple pendulum. Here the tension of the string is always at right angles to the direction in which the mass at the end of the string is moving, and therefore that force does no work.

105. Units of Work.

The unit of work in use in scientific measurements is the **erg**, which is defined to be the work done by the force of a dyne acting through a centimetre. The corresponding unit of work in the British system of units is the **foot-poundal**, the work done by the force of a poundal acting through the distance of a foot. Since the C.G.S. system of units has replaced the British system in scientific measurements, the foot-poundal is rarely used as a unit of work.

The unit of work adopted in practice by engineers is a gravitational one. In this country it is the **foot-pound**, which is the work done by a force equal to the weight of one pound acting through the distance of one foot. We have seen that the weight of one pound is not the same at all points on the earth's surface, and therefore the foot-pound is not an invariable unit. As, however, the variation in the weight of one pound is extremely small, no inconvenience results in practice from adopting the foot-pound as a unit of work.

EXAMPLE 1.—Find the work done in foot-pounds in lifting 15 cwts. of coal from a mine 120 fathoms deep.

The force required to lift the 15 cwts. is of course the weight of that mass, that is, is the force of 15×112 lbs., and this force acts through the distance of 120 fathoms or 120×6 feet. Hence the work done is

$$112 \times 15 \times 120 \times 6 = 1209600 \text{ foot-pounds.}$$

EXAMPLE 2.—How many ergs are in a foot-pound at a place where $g=981$? Foot=30·4797 centimetres; pound=453·59 grammes.

The weight of m grammes is mg dynes, and therefore the weight of 1 pound, or 453·59 grammes, at a place where $g=981$, is

$$453{\cdot}59 \times 981 \text{ dynes.}$$

Since a foot is equivalent to 30·4797 centimetres, the weight of one pound acting through a foot is equivalent to the force of $453{\cdot}59 \times 981$ dynes acting through 30·4797 centimetres, and is therefore equivalent to the force of $453{\cdot}59 \times 981 \times 30{\cdot}4797$ dynes acting through the space of one centimetre. Hence at a place where $g=981$,

$$\begin{aligned} \text{foot-pound} &= 453{\cdot}59 \times 981 \times 30{\cdot}4797 \text{ ergs} \\ &\doteq 13570000 \text{ ergs nearly.} \end{aligned}$$

106. Effect of Work done on a Body.

When a force acts on a body whose motion is not resisted by the action of other forces, the body moves in the straight line in which the force acts with a constant acceleration. In such a case the whole effect of the work done by the force on the body is to increase the velocity of the body.

Let u be the velocity of a body of mass m when a force f begins to act. By Newton's second law of motion the force f causes an acceleration $\frac{f}{m}$ in the body, and therefore if v be the velocity when the body has passed over the space s, by Article 27

$$v^2 = u^2 + 2\frac{f}{m}s, \text{..........................}(1)$$

Multiplying every term in this equation by $\frac{1}{2}m$, we get

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mu^2 + fs, \text{........................}(2)$$

Equation (1) shows that the increase in the square of the velocity, $v^2 - u^2$, is

$$2\frac{fs}{m}.$$

Equation (2) is sometimes referred to hereafter as the "equation of energy."

When a force acts on a body whose motion is resisted by the action of other forces, the increase in the velocity for a given quantity of work will be less than in the former case, as some of the work will be spent in overcoming the resistances. When the force acting on a body is just sufficient to overcome the resistances, the whole of the work done by the force is spent in overcoming the resistances and there is no increase in velocity due to the action of the force. Thus in the first example of the preceding Article the force necessary to raise the cage is a force equal to the weight of the cage and its contents, and if the tension of the rope by which the cage is raised be just equal to the weight of the cage, we have a case in which the force acting on the body is resisted by an equal force, and therefore the effect of the work done on the body is to move the body against the resistance without any increase in velocity. As another example take the case of a horse dragging a wagon at a uniform rate up a hill. Here the work done by the tension of the traces is spent in overcoming the component of the force of gravity down the hill, the forces of friction between the ground and the wheels, and between the axles and the wheels, and lastly the resistance of the atmosphere. Since there is no increase in velocity the whole work is spent in moving the wagon against these resistances.

107. Energy.

The **energy** of a body is the work that can be got from a body in virtue either of its motion or of its position.

That work can be obtained from a body in motion is evident from observations of everyday life. Thus, for example, a shot fired from a heavy gun has the property in virtue of its motion of overcoming a great resistance, and may penetrate through several inches of solid iron. A curling-stone projected along the ice does work against the forces of friction and the resistance of the atmosphere.

That work may be obtained from a body in virtue of its position is seen in the case of many familiar examples. Thus, in the case of a pile-driver, the ram, a massive lump of iron, is raised several feet above the top of the pile which is to be driven in. In this position the ram has no energy due to motion, but it has energy due to its position above the pile. On being released from this position it begins to acquire velocity which continually increases, so that its energy due to motion is ever increasing. At the moment when the ram strikes the top of the pile the whole of the energy due to position has been converted into energy of motion, and this energy is suddenly spent in driving in the pile into the ground against resistance. We see then that work may be obtained from a body on account of its motion or on account of its position.

The energy of a body due to motion is called **Kinetic Energy**, the energy due to position is called **Potential Energy.**

It is evident that the energy of a body is a measurable quantity, and that it is a quantity of the same kind as work. It is therefore to be measured in the same units as work.

If a body of mass m is moving with a velocity v, the kinetic energy in absolute units is $\frac{1}{2} m v^2$.

EXAMPLE 1.—Calculate in foot-pounds the energy of a train of 100 tons moving with a velocity of 30 miles an hour.

A velocity of 30 miles an hour is the same as a velocity of 44 feet per second, and a mass of 100 tons is equal to the mass of $100 \times 20 \times 112$ lbs. Hence the energy of the train in foot-poundals is

$$\tfrac{1}{2} \times 100 \times 20 \times 112 \times 44^2,$$

and the energy in foot-pounds is (taking $g = 32$)

$$\frac{100 \times 20 \times 112 \times 44^2}{2 \times 32} = 6776000.$$

EXAMPLE 2.—A body whose mass is m grammes is shot vertically upwards with a velocity of v_1 centimetres per second, and on returning to the same point has a velocity of v_2 centimetres per second; due to the resistance of the atmosphere v_2 is less than v_1. Calculate in ergs the loss of energy.

The body leaves the ground with a velocity v_1, and hence its kinetic energy at the moment of projection is $\frac{1}{2} m v_1^2$. It returns to the ground with a velocity v_2, or with a kinetic energy of $\frac{1}{2} m v_2^2$. The body in moving through the air does work against the resistance of the air, and this is shown by a loss of energy. The amount of this loss is

$$\tfrac{1}{2} m v_1^2 - \tfrac{1}{2} m v_2^2 = \tfrac{1}{2} m (v_1^2 - v_2^2) \text{ ergs.}$$

108. Connection between Work and Energy.

We saw in Article 106 that if a force does work on a body whose motion is unresisted, the effect of the work done is to increase the velocity of the body. From equation (2) of that Article it follows that the work done on the body is equal to the increase in the kinetic energy of the body. For f being the force, s the space through which it acts, m the mass of the body, and u and v the initial and final velocities respectively, fs represents the work done on the body, and $\frac{1}{2} m v^2 - \frac{1}{2} m u^2$ is the increase in the kinetic energy of the body. Equation (2) shows that these are equal. Hence when the motion of a body is unresisted the amount of work done on the body is equal to the increase in the kinetic energy of the body.

When a force does work on a body whose motion is resisted by the action of other forces, the increase in the kinetic energy of the body is less than if the motion were unresisted by the amount of work which is done in overcoming these resisting forces. In some cases the whole of the work which is done on the body may be spent in doing work against the opposing forces. In such a case there will be no increase in the kinetic energy, and the amount of work done will be equal to the increase of the potential energy.

As an illustration of the case in which the whole of the work is spent in increasing the kinetic energy, take the case of a body of mass m falling from a height h. The force acting on the body is the weight $m\,g$, and the distance through which this force acts is h, so that the work done on the body is mgh. Now if we neglect the resistance of the atmosphere, the body after falling through a height h will have a velocity v, given by the equation,

$$v^2 = 2gh.$$

The kinetic energy, therefore, or $\frac{1}{2}mv^2$, will be equal to mgh, that is, the kinetic energy will be equal to the work done on the body.

As an illustration of the case in which the work done on the body is spent in overcoming resistances, we may take the case of a mass lifted steadily through a height. In this case the work done on the body is done in overcoming the resistance, the weight of the body, and the equivalent of the work done on the body is the increase of the potential energy of the body, that is, the energy of the body due to the height above the surface of the ground. The amount of this potential energy is the amount of work that is stored in the body in virtue of its position. If m be the mass of the body and h the height through which the body has been lifted, the work which can be obtained by allowing the body to fall is evidently mgh, and this is the amount of work which has been done on the body in lifting it to the height h against the force mg.

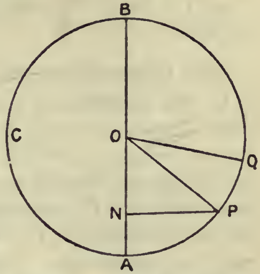
109. Conservation of Energy.

In the simple illustrations given in the preceding Article of the relation between the work done on a body by a single force and the increase in the energy of the body, the amount of work done was shown to be equal to the increase in the energy. The doctrine of conservation of energy asserts that in every case in which work is done on a body by a system of forces there is always an equivalent increase in the energy of the body. This energy may either be wholly kinetic or wholly potential, or partly kinetic and partly potential, but in every case the sum of the kinetic and potential energies generated is equal to the amount of work which is done on the body by the system of forces. Conversely, if a body is losing energy we infer that work is being done against resistances, and that the loss of energy is equal to the amount of work which is being done. The doctrine may be stated shortly thus:—Energy cannot be destroyed and cannot be generated. It may be communicated from one body to another, and it may pass from the kinetic form to the potential form, or *vice versa*, but it is as indestructible as matter itself.

The doctrine of the conservation of energy appears to be contradicted by innumerable phenomena which we see in everyday life. A railway train comes to rest when the steam is shut off and the brakes applied; a billiard ball struck by a cue comes at length to rest; a stone projected vertically upwards returns to the ground with less kinetic energy than when it started. In each of these cases there is apparently loss of energy, but it is to be noticed that in each of these cases work is done against the force of friction, and that this work, although apparently lost, appears again in the form of heat, which is a form of energy. The consideration of the energy which is apparently lost when a body moves against the force of friction belongs more to physics than to dynamics. We shall therefore only remark that, when we take into account the work done against frictional and other resistances, the doctrine of conservation of energy has been found, in innumerable experiments, to be absolutely true.

110. Motion of a Body in a Vertical Circle.

A body, considered as a particle, is suspended by a string from a point O, and moves in a vertical circle APB. Required to find the relation between the velocity of the body when passing through the lowest point A of the circle, and the velocity when passing through any other point P.



Let V denote the velocity at A,

v „ „ „ P,

r „ „ radius of the circle,

θ „ „ angle AOP.

The kinetic energy at A is $\frac{1}{2}mV^2$, m being the mass of the body; and the kinetic energy at P is $\frac{1}{2}mv^2$. Hence the loss of kinetic energy after passing over the arc AP is

$$\tfrac{1}{2}m(V^2 - v^2).$$

The gain in potential energy is the weight multiplied by the height through which the body has risen. If PN be drawn perpendicular to OA, the gain in potential energy is

$$\begin{aligned} &= mg\text{AN} = mg(\text{OA} - \text{ON}) \\ &= mg(\text{OA} - \text{OP}\cos\theta) = mgr(1 - \cos\theta). \end{aligned}$$

Now by the principle of conservation of energy the gain in potential energy is equal to the loss of kinetic energy. Hence

$$\tfrac{1}{2}m(V^2 - v^2) = mgr(1 - \cos\theta),$$

from which

$$V^2 - v^2 = 2gr(1 - \cos\theta).$$

Corollary 1.—If the body comes to rest at a point Q, where the angle AOQ is α, we put $v = 0$ and $\theta = \alpha$ in the above equation. We thus obtain

$$V^2 = 2gr(1 - \cos\alpha).$$

If the angle α can be found by observation, this formula will enable us to find the velocity V at the lowest point.

Corollary 2.—If the body just reaches the highest point B, we put $v = 0$ and $\theta = \pi$ in the equation of energy. We thus obtain (since $\cos\pi = -1$),

$$V^2 = 4gr,$$

which gives

$$V = 2\sqrt{gr}.$$

After reaching the point B, the body will, of course, fall down the arc BCA, passing through A with the same velocity as before.

111. Horse-power.

In the definition of work given in Article 104 the idea of time is not involved. In actual practice, however, the question of the rate at which an agent works is an important one, and engineers classify agents according to the rate at which they do work. It was estimated by James Watt, the improver of the steam-engine, that a horse can do 33,000 foot-pounds of work in one minute. This estimate is considered now to be too high, but an agent which does that number of foot-pounds of work in one minute is still said to be of one horse-power. An agent which does half that number of foot-pounds of work in one minute is said to be of one-half horse-power, and an agent which does ten times that number of foot-pounds of work in one minute is said to be of ten horse-power; and so on.

Example.—If a pressure of one ton is exerted through a space of 10 yards, how many foot-pounds of work are done, and at what horse-power does an engine work which does that amount of work in half a minute?

$$1 \text{ ton} = 112 \times 20 \text{ pounds};$$
$$10 \text{ yards} = 30 \text{ feet};$$

hence the number of foot-pounds of work done is

$$30 \times 112 \times 20 = 67200.$$

If an engine does this amount of work in half a minute, it will do twice that number of foot-pounds, or 134400, in one minute. Hence the horse-power is

$$\frac{134400}{33000} = 4\tfrac{4}{55}.$$

EXAMPLES XXVI.

1. A 6-lb. cannon-ball is discharged with a velocity of 1600 feet per second. How many foot-pounds of work have been done upon the ball by the exploding powder, and what has been the average force of propulsion if the barrel is $3\frac{1}{2}$ feet long?

   What horse-power is represented by the discharge of 20 such balls per minute?

2. A bullet whose mass is $2\frac{1}{4}$ oz. leaves the mouth of a gun with a velocity of 1550 feet per second. The length of the gun barrel is $2\frac{1}{2}$ feet. Find the average force upon the bullet in poundals.

3. A 10-ton hammer falls through a height of 6 feet, and makes an impression on a mass of iron to the extent of 1 inch. Find the average pressure on the mass of iron in pounds' weight which has been exerted during the blow.

4. A particle whose mass is 27 lbs. moves from rest under the action of a constant force which does on it 75 foot-pounds of work in a certain time. What is the velocity of the particle at the end of that time?

5. A hole is punched through a plate of iron one half-inch in thickness, and the pressure actuating the punch is estimated at 360,000 lbs. weight. Assuming the resistance to be uniform, find the number of foot-pounds of work done.

6. Explain how it is that we can easily drive in a nail into a piece of wood by striking it with a hammer, although we may not be able to drive it in by merely pressing the hammer against it.

7. A mass of 1000 lbs., moving with a velocity of 500 feet per second, meets with an obstacle. What is the energy of the blow?

   If the obstacle yield one inch, what is the average force of the blow?

8. Compare the momenta and also the kinetic energies of two unequal masses m and m' when acted on by equal forces—(1) for equal times, (2) over equal distances.

9. A shell moving at the rate of 1000 feet per second explodes symmetrically into ten 2-lb. fragments. One of these has its velocity increased in magnitude without any change of direction. Find its new velocity, assuming that 250,000 foot-pounds of work are generated by the explosion.

10. A pumping-engine is partly worked by a weight of two tons, which at each stroke of the pump falls through 4 feet, and the pump makes 10 strokes a minute. Find how many gallons of water—one gallon weighing 10 lbs.—are lifted per minute from a depth of 200 feet.

    [The question of how the weight is lifted between the strokes need not be considered.]

11. An endless cord, stretched and running on grooved pulleys with a velocity of 300 feet per minute, transmits 5 horse-power. Find the tension of the string.

12. A train whose mass is 100 tons is drawn at the rate of 40 miles an hour along a railway. If the resistance is 8 lbs. per ton, find the horse-power at which the engine is working.

13. The diameter of the steam cylinder of an engine is 3 feet 4 inches, and the length of the stroke of the piston is 8 feet. If the mean pressure of steam on the piston is 35 lbs. per square inch, at what horse-power is the engine working when the piston is making 12 strokes per minute?

14. A man weighing 12 stones does 360,000 foot-pounds of work against gravity in ascending a hill. Find the height ascended.

15. A body whose mass is 10 lbs. is placed on an inclined plane, inclination 30°, and is made to slide up a distance of 50 feet. What number of units of work is done by the force?

If the coefficient of friction between the body and the plane is ·3, what number of units of work is done against friction?

16. A particle whose mass is m moves from rest under the action of a force which does on it U units of work in a certain time. What is its velocity at end of that time?

Suppose the work $=75$ foot-pounds ($g=32$), and the mass of the body is 27 lbs., what is its velocity?

17. A body whose mass is 10 lbs. slides down an inclined plane whose height is 25 feet. It reaches the foot of the plane with a velocity of 30 feet per second. How many foot-pounds of work have been expended during the motion on friction and other resistances?

CHAPTER X.—IMPULSIVE FORCES.

112. Impulsive Forces.

An **impulsive force**, or shortly an **impulse**, is an infinitely great force acting for an infinitely short time.

As an illustration of an impulse consider the action between a cricket-ball and the bat during a "drive." Before the ball is struck by the bat, it is moving with a certain velocity in a certain direction, and immediately after being struck it is moving with a different velocity in another direction. This change in the motion of the ball is due to the force exerted by the bat on the ball during the infinitely short space of time during which the ball was in contact with the bat. This is a case in which a finite change of velocity is produced in an infinitely short space of time, and therefore the force which causes that change must be infinitely great. The action between the ball and the bat is an example of an impulse.

A finite force acting on a body for an infinitely short time would produce an infinitely small change of velocity. The action between the cricket-ball and the bat is completed in probably less than the ten-thousandth part of a second, and yet during that time a finite change of velocity is produced. The change of velocity which would be produced by the force of gravity in the same time would only be ·0032 foot per second.

Impulses are to be measured by the changes of momentum which they produce.

113. Since action and reaction are equal and opposite, when an impulse acts between two bodies, the force on one of the bodies must be equal and opposite to the force on the other body. Hence, by the second law of motion, the change of momentum produced in one body must be equal and opposite to the change of momentum produced in the other. Thus when a cannon is fired the momentum of the shot when leaving the mouth of the gun must be equal and opposite to the momentum with which the gun recoils.

Example.—A 40-lb. shot is fired from a 12-ton gun with a velocity of 800 feet per second. What is the velocity with which the gun recoils?

Taking one pound as the unit of mass and the velocity of one foot per second as the unit of velocity, the momentum of the shot is $40 \times 800 = 32000$. If v be the velocity of the recoil, the momentum of the gun is $12 \times 20 \times 112\, v$. This is equal to the momentum of the shot, and therefore

$$12 \times 20 \times 112\, v = 32000,$$

from which $$v = \frac{25}{21} \text{ feet per second.}$$

114. Impact or Collision of Bodies.

When two bodies come into collision the time during which they are in contact is infinitely short. We may, however, divide this time into two periods. In the first period of the impact the two bodies are coming into closer contact with each other, each body being compressed, due to the action of the other body on it. As long as compression is going on the velocities of the two bodies are becoming more and more nearly equal, and at the instant when the compression has reached its greatest amount the two bodies are moving together with the same velocity, just as if they formed parts of the same rigid body. After the period of compression comes the period of restitution. Every body in nature possesses **elasticity**, a property in virtue of which it resists change of shape and tends, when deformed, to return to its original shape. In the first period of the impact the compression is resisted by the elasticity of the two bodies, and in the second period the same property causes the bodies to return to their original shapes. Hence during the second period the two bodies are in the act of separating from each other, and at the end of the impact they move away from each other.

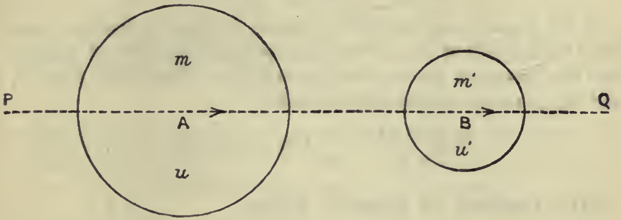
We shall consider only two simple cases of impact, one in which two smooth spherical balls moving in the same straight line come into collision with each other, and the other in which a ball strikes a plane surface perpendicularly.

115. Impact of Inelastic Balls.

Every body in nature possesses some elasticity, but there are some bodies, such as soft clay, whose elasticity is very small. In this Article we consider the abstract problem of the impact of two perfectly inelastic balls.

Let A and B be two smooth, spherical, inelastic balls moving with their centres in the straight line PQ. We shall take as the typical case the case in which both balls are moving in the same direction PQ before impact. Let m and m' be the masses, u and u' the velocities of A and B respectively, u being greater than u', so that A is overtaking B.

Since the two balls are perfectly inelastic they will not separate after the impact, but will move together with the same velocity as if they were parts of a rigid body. Let v be their common velocity after impact, then v will be found by



considering that during the collision the action of A upon B is equal and opposite to the action of B upon A. Hence the momentum which is gained by B must be equal to the momentum which is lost by A. The momentum gained by B is

$$m'v - m'u';$$

the momentum lost by A is

$$mu - mv;$$

and these two expressions are equal. Hence

$$mu - mv = m'v - m'u',$$

from which
$$v = \frac{mu + m'u'}{m + m'},$$

an equation giving their common velocity after impact in terms of the masses and the velocities before impact.

We can easily adapt this formula to the case in which the two bodies are moving before impact in opposite directions, by considering one of the velocities as positive and the latter as negative.

Example.—Two inelastic balls, whose masses are 8 lbs. and 15 lbs., are moving with velocities of 6 feet per second and 3 feet per second respectively. Find the common velocity after impact if the balls are moving before impact, (1) in the same direction, (2) in opposite directions.

Here $m = 8,\ m' = 15.$

If the balls are moving before impact in the same direction, $u=6$, $u'=3$. Hence the common velocity after impact is

$$\frac{6\times8+3\times15}{8+15}=\frac{93}{23} \text{ feet per second.}$$

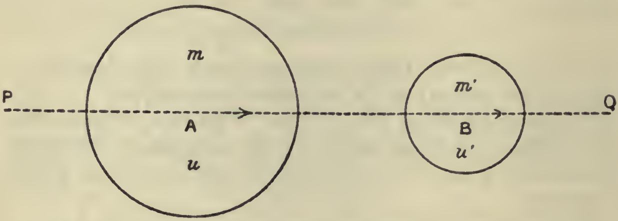
If the balls are moving before impact in opposite direction, consider the velocity of one of the balls (say the 8-lb. ball) as positive and the velocity of the other ball as negative. Then $u=6$, $u'=-3$, and the common velocity after impact is

$$\frac{6\times8-3\times15}{8+15}=\frac{3}{23} \text{ foot per second.}$$

116. Impact of Elastic Balls.

Newton found by experiment that in the impact of two elastic balls the relative velocity after impact always bore a certain ratio to the relative velocity before impact, the value of this ratio depending only on the nature of the substances of which the balls were composed. This ratio is called the **coefficient of restitution** for the two given substances. Newton's experiments are described in Article 118.

Let A and B be two smooth, spherical, elastic balls moving with their centres in the same straight line PQ. As in the preceding Article, we take as the typical case the case in which the two balls are moving in the same direction. Let m and m' be the masses of the balls A and B respectively, u and u' their velocities before impact; u being greater than u', so that A is overtaking B. Let v and v' be the velocities of A and B respectively after impact. The relative velocity before impact, that is, the velocity with which A is approaching B, is $u-u'$.



The relative velocity after impact, that is, the velocity with which the two balls are separating, is $v'-v$. If e denote the coefficient of restitution between the two balls, then

$$v'-v=e\,(u-u') \qquad \text{(1)}$$

Also, as in the preceding Article, there is no loss of momentum due to the impact. Hence

$$m'v'+mv=m'u'+mu \qquad \text{(2)}$$

Equations (1) and (2) are two simultaneous equations, from which we can find v and v' in terms of m, m', u, u' and e. Solving these simultaneous equations we obtain

$$v'=\frac{mu+m'u'}{m+m'}+\frac{me\,(u-u')}{m+m'} \qquad \text{(3)}$$

$$v=\frac{mu+m'u'}{m+m'}-\frac{m'e(u-u')}{m+m'} \qquad \text{(4)}$$

If $e=0$, that is, if the balls are supposed to be perfectly inelastic, the values of v and v' given by equations (3) and (4) become equal, their common value being

$$\frac{mu+m'u'}{m+m'},$$

as found in the preceding Article.

For all bodies in nature the value of e is a proper fraction, but the student may meet with examples of the abstract case in which e is supposed equal to unity. The two balls are then said to be *perfectly elastic.*

Example.—Determine the velocities after impact of two balls whose masses are 7 lbs. and 13 lbs. respectively moving in the same straight line, with velocities of 5 feet per second and 3 feet per second respectively, (1) when they are moving in the same direction, (2) when they are moving in opposite directions. $e=\frac{1}{2}$.

Here $m=7,\ m'=13$.

(1) If the balls are moving in the same direction, $u=5$, $u'=3$. Let v and v' denote the velocities after impact; then

$$v'-v=\tfrac{1}{2}\,(5-3)=1,$$

and

$$\begin{aligned}13v'+7v&=5\times 7+3\times 13\\&=74.\end{aligned}$$

Solving these two equations, we find

$$v' = \frac{81}{20} = 4\tfrac{1}{20} \text{ feet per second,}$$

$$v = \frac{61}{20} = 3\tfrac{1}{20} \text{ feet per second,}$$

the velocities of the 13-lb. and the 7-lb. balls respectively.

(2) If the balls are moving in opposite directions, we consider the direction of the 7-lb. ball as the positive direction. We have then $u=5$, $u'=-3$, and if v and v' denote the velocities after impact we have the two equations

$$\begin{aligned} v' - v &= \tfrac{1}{2}\{5-(-3)\} \\ &= 4, \\ 13v' + 7v &= 5 \times 7 - 3 \times 13 \\ &= -4, \end{aligned}$$

from which

$$v' = \frac{24}{20} = \tfrac{6}{5} \text{ foot per second,}$$

$$v = -\frac{56}{20} = -\tfrac{14}{5} \text{ feet per second.}$$

Since v is negative, the 7-lb. ball will have its direction of motion reversed by the collision.

117. Impact of an Elastic Ball on a Fixed Smooth Plane Surface.

We suppose the ball to be moving before impact in a direction at right angles to the surface. Then, since the action between the ball and the surface is perpendicular to the surface, the ball will rebound from the surface in the straight line in which it was moving before impact.

Let u be the velocity before impact, and v the velocity after impact. Then we may consider the surface as a ball of mass so great compared with the mass of the impinging ball that its velocity is not altered by the impact. Hence we may put $u'=0$ and $v'=0$ in equation (1) of Article 116. We thus obtain

$$-v = eu \quad \text{or} \quad v = -eu.$$

This gives the velocity after impact, the negative sign showing that the velocity is reversed in direction.

118. Newton's Experiments.

Newton's experiments for determining the coefficient of restitution between two balls of given materials were performed in the following manner:—

The two balls A and B were suspended by strings of equal length from points C and D in the same horizontal line. Fig. 1 shows the positions of the balls at rest, the strings AC and BD being vertical.

One of the balls B is then drawn aside into such a position E as is shown in Fig. 2, and the angle EDF which ED makes with the vertical was measured. This ball B was then let go, and came into collision with A at rest, the action between the two bodies being in the line joining the centres of the two balls, that is, in the line in which B is for the moment moving. The balls after impact described arcs of circles before coming to rest, and the angles of these arcs, HDG and KCL in Fig. 3, were measured.

Fig.1.

Knowing the angle EDF, we can find by Article 110 the velocity of B at the moment when it comes into collision with A, and knowing the angles HDG and KCL through which the strings rotate after impact, we can calculate by the

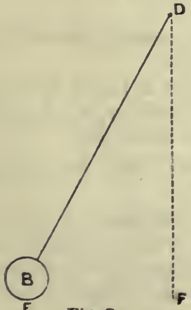


Fig. 2.

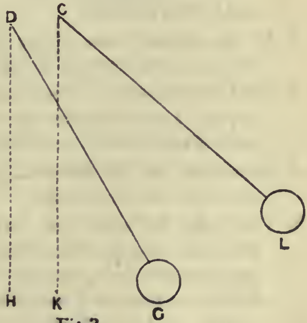


Fig. 3.

same Article the velocities of B and A immediately after the impact.

From experiments performed in this way Newton deduced the result that the relative velocity after the impact of two balls of given materials always bears a fixed ratio to the relative velocity before impact. This fixed ratio is now called the coefficient of restitution.

EXAMPLES XXVII.

1. The mass of a gun is 8 lbs. and of the shot $1\frac{1}{2}$ oz. When the gun is fired the initial velocity of the recoil is 16 feet per second. What is the initial velocity of the shot?

2. Two inelastic balls whose masses are 15 lbs. and 17 lbs. are moving in the same straight line with velocities of 17 feet per second and 15 feet per second. Find their velocities after impact, if before impact they are moving, (1) in the same direction, (2) in opposite directions.

3. Two balls for which the coefficient of restitution is $\frac{2}{3}$ are moving in the same straight line. The masses of the two balls are 13 lbs. and 11 lbs., and they are moving in opposite directions with velocities of $3\frac{1}{4}$ feet per second and $4\frac{1}{3}$ feet per second respectively before impact. Find their velocities after impact.

Calculate in foot-poundals the amount of energy which is lost by the collision.

4. In the abstract case of the balls being perfectly elastic, show that no kinetic energy is lost by collision.

5. Find the amount of kinetic energy which is lost in the direct impact of two inelastic balls of masses m and m', moving in the same direction with velocities u and u' respectively.

6. Show that in the direct impact of two elastic balls, kinetic energy is lost by collision.

7. Two balls, for which the coefficient of restitution is $\frac{3}{4}$, are in the same straight line. The masses of the two balls are 25 grammes and 125 grammes. Before impact the 125-grammes ball is at rest, and the other is moving towards it with a velocity of 45 centimetres per second. Find the velocities of the balls after impact.

Calculate in ergs the amount of energy which is lost by the collision.

8. One perfectly elastic ball impinges on another perfectly elastic ball at rest. What must be the ratio of the masses of the two balls in order that the first may lose an nth part of its velocity by the impact?

9. A ball let fall on to a stone slab from a height of 16 feet bounces the first time to a height of 9 feet. What is the coefficient of restitution, neglecting the resistance of the air?

Find also the total distance it will travel before coming to rest.

10. A ball falls vertically under the action of gravity through a height h, and after rebounding from a horizontal plane rises again to a height K. Determine the coefficient of restitution in terms of h and K.

11. A ball falling vertically strikes the ground and rebounds. If the coefficient of restitution is $\frac{1}{3}$, find how much of its kinetic energy the body loses. Explain what becomes of this energy.

12. A ball A impinging directly on a ball B at rest, is itself brought to rest. Show that, e being the coefficient of restitution,

A's mass $= e \times$ B's mass,
B's velocity after impact $= e \times$ A's velocity before impact.

CHAPTER XI.—MACHINES.

119. Definition of a Machine.

A **machine** is an appliance for the transference of energy from one point to another.

This definition is illustrated in one of the commonest machines, the steam-engine. In the steam-engine the steam which enters the cylinder possesses energy in virtue of its tendency to expansion, and part of this energy is spent in doing work in maintaining the motion of the piston. The energy of the steam is thus communicated to the piston, and by proper appliances this energy can be transferred to some other point. Thus in the case of the mining engine the energy is transferred to a rope by means of which work is done against the force of gravity in lifting coals from the bottom of the mine to the surface. The steam-engine therefore is an appliance which receives at one point a supply of energy which is spent in doing work at some other point.

In all machines some of the energy which is supplied to the machine is apparently lost in the transference, the machine giving out less energy than is supplied to it. This apparent loss is due principally to the work which is done against the friction between the different parts of the machine. Now work done against friction is always followed by the appearance of heat, which is a form of energy. It is found by experiment that a definite amount of work is followed by the appearance of a definite amount of heat, and that the energy which disappears in the machine has its equivalent in the amount of heat generated. We may say, however, that the work which is spent in doing work against friction in a machine is practically lost, as it reappears in a form which is not available for doing useful work.

The steam-engine is a much more complicated machine than any we shall have to deal with in this chapter. The machines which we are about to consider are simple in construction and are supposed, unless it is explicitly stated to the contrary, to work without friction. In each case the machine is employed in the following way:—At some given point of the machine a resistance is to be overcome—the resistance is often a weight to be raised against the force of gravity—by means of a force acting at some other given point of the machine. The resistance to be overcome is generally called the **weight**, and the force necessary to overcome it is called the **power**. The general problem in machines is to find the relation between the power and the weight, and is solved by the application of the principles of statics. If, as is usually done, we denote the weight by W and the power by P, we have to find by statics the magnitude of a force P which will be required to act at a given point of a machine to balance another given force W acting at some other given point of the machine. The machine will remain at rest under the action of P and W, and it would require a small force additional to P to overcome the inertia of the machine and set it in motion. It is usual, however, in abstract dynamics to suppose that the force P found as above is sufficient to set the machine in motion. Work is done on the machine by the power P acting through a certain distance, and this work is given out by the machine in moving the weight W through a corresponding distance.

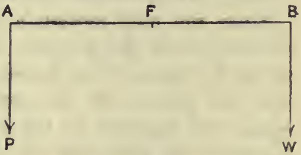
The **mechanical advantage** of a machine is the ratio of the weight to the power, and is therefore measured by the fraction

$$\frac{W}{P}.$$

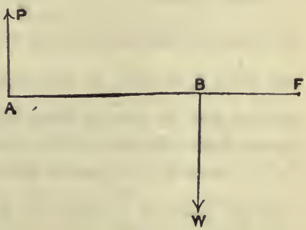
120. The Lever.

The typical form of the lever is a straight rod free to turn in one plane about a fixed point called the fulcrum. Levers are generally classified according to the relative positions of the power, the weight and the fulcrum.

Class I.—Fulcrum between the power and the weight.

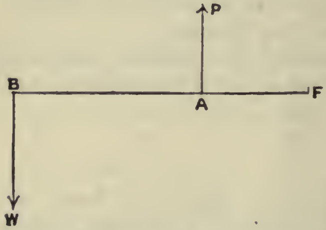


A lever of this class is shown in the diagram. F is the fulcrum which is between A and B, the points where the power and the weight act respectively. Examples of levers of this class are seen in the treadle of a lathe, in the beam often used on canals for the purpose of drawing wooden bridges. Another example of a lever, though differing from the typical form, is seen in the lock gate of a canal. The fulcrum is the pivot round which the gate turns, the weight is the resistance experienced in moving the gate against the pressure of the water, while the power is supplied at the end of the beam attached to the lock gate. A pair of scissors is an example of a double lever of this class.



Class II.—Weight between the fulcrum and the power.

Examples of levers of this class are: (1) a crowbar with one end resting on the ground, used for lifting a weight or moving it along the surface of the ground; (2) the lever in a Bramah's press (see Article 134). A pair of nut-crackers is a double lever of this class.



Class III.—Power between the fulcrum and the weight.

When a weight is lifted in the hand the forearm acts as a lever of this class. The fulcrum is at the elbow joint, the weight is in the hand, and the power is the tension of the tendons connecting the forearm with the biceps, the large muscle of the upper part of the arm. A pair of tongs is a double lever of this class.

121. Mechanical Advantage of a Lever.

Since the only motion a lever can have is a motion of rotation round the fulcrum, the sufficient and necessary condition for equilibrium between the power and the weight is that the moment of the power round the fulcrum shall be equal and opposite to the moment of the weight. In the figures of the preceding Article the power and the weight are both supposed to act perpendicular to the lever; but, whatever be their directions, let P and W represent the power and the weight, and a and b the distances of their lines of action respectively from the fulcrum. Then by the principle of moments

$$Wb = Pa,$$

from which

$$\frac{W}{P} = \frac{a}{b}.$$

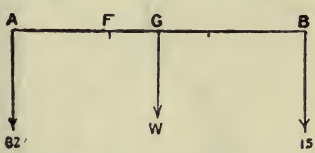
In levers of the first class a may be either greater, equal to, or less than b, so that in that class of levers the fraction $\frac{W}{P}$ may or may not be greater than unity. Hence in the first class of levers there may be or there may not be mechanical advantage.

In levers of the second class a is greater than b, and therefore the fraction $\frac{W}{P}$ must be greater than unity, that is, there must be mechanical advantage.

In levers of the third class a is less than b, and therefore the fraction $\frac{W}{P}$ must be less than unity, that is, there cannot be mechanical advantage.

The student will notice that in finding the relation between the power and the weight we have neglected the weight of the lever. If the weight of the lever cannot be neglected, we find the relation between the power and the weight by expressing the condition that the algebraical sum of the moments of the weight of the lever, the power, and the weight round the fulcrum is zero.

Example.—A uniform lever balances about a point $\frac{1}{3}$ of its length from one end when 32 lbs. are hung from this end and 15 lbs. from the other. Find the weight of the lever.



Let AFGB be the lever, F being the fulcrum. $AF = \frac{1}{3} AB$, $FB = \frac{2}{3} AB$, and since the rod is uniform the C.G. of the lever will be at G, the middle point of AB. Hence $FG = AG - AF = \frac{1}{2} AB - \frac{1}{3} AB = \frac{1}{6} AB$.

Let W denote the weight of the lever; then by the principle of moments the algebraical sum of the moments round F of 32 at A, 15 at B, and W at G is zero. Hence

$$W \times FG + 15 \times FB = 32 \times FA,$$

or

$$W \times \tfrac{1}{6} AB + 15 \times \tfrac{2}{3} AB = 32 \times \tfrac{1}{3} AB,$$

or, dividing by AB, and clearing of fractions,

$$W + 60 = 64,$$

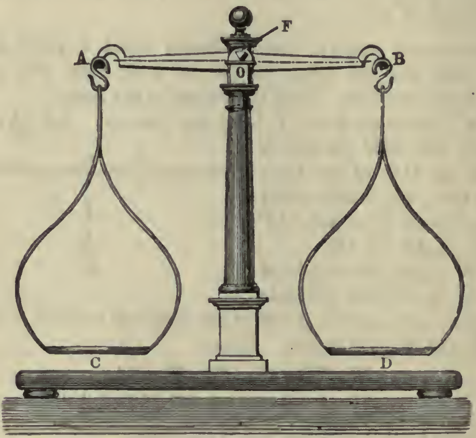
from which

$$W = 4 \text{ lbs.}$$

122. The Balance.

The balance in its simplest form is a uniform lever AB, called the beam, with similar scale-pans C and D suspended one from each end of the lever, the whole being pivoted on a fulcrum F midway between the two ends of the beam. The C.G. of the balance will evidently be in the vertical line through F, and any mass placed in one scale-pan will balance an equal mass placed in the other scale-pan.

The student must carefully notice that such a balance measures masses and not weights. In the process known as weighing a body in such a balance we place the body in one scale-pan C, and find by trial what mass we must place in the other scale-pan D, in order to balance the given body. Suppose we find that we must place seven masses, each of 1 lb., in scale D, then we say that the body in scale C has a mass of 7 lbs. We cannot tell from this measurement what is the weight of the given body, because the weight of the body is the force with which the body is attracted to the earth's surface, and this weight is equal to the weight of the equal mass in the other scale-pan, whatever be the magnitude of the force of gravity. Even though the force of gravity were diminished or increased a thousand fold, the body would still be balanced by a mass of 7 lbs. placed in the scale-pan D. In fact the force of gravity *is* different at different points of the earth's surface, and still a body requires the same mass to balance it at whatever part of the earth the measurement is made. On account of this, the *mass* of a body is often called its *weight*.



123. False Balance.

If the parts of the beam on either side of the fulcrum are of unequal length, the balance will be false, and will give a false indication for the mass of a body. A correct determination of the mass of a body can be found from a false balance by means of the process known as double weighing. The body to be weighed is placed in one scale-pan, and fine sand is then poured into the other scale-pan until an exact balance is obtained. The body is then removed from the scale-pan, and in its place is put standard masses, pounds, ounces, &c., sufficient to balance the sand in the other scale-pan. It is evident that whether the balance is true or false, the mass of the body must be exactly equal to the standard masses put in to balance the sand.

Another method of obtaining a correct result with a balance of unequal arms is to proceed as follows:—We place the body in one scale-pan and balance it against standard masses in the other scale-pan. Removing the body and the standard masses, we place the body in the other scale and again balance it by standard masses. It is evident that the two determinations will be different. Taking the figure of the preceding Article, let a and b denote the lengths of the two parts FA and FB of the beam; also let W_1 be the apparent weight when the body is weighed in the scale C, W_2 the apparent weight when weighed in the scale D, and W the true weight. Then by the principle of moments, the moment round the fulcrum of W acting at A must be equal and opposite to the moment of W_1 acting at B. Hence

$$Wa = W_1 b.$$

Similarly

$$Wb = W_2 a.$$

Multiplying the two equations together we get

$$W^2 ab = W_1 W_2 ab,$$

or, dividing by ab, and extracting the square root,

$$W = \sqrt{(W_1 W_2)}.$$

Hence the true weight is the geometric mean of the two apparent weights when weighed in the two scale-pans.

To find the ratio of the arms in terms of W_1 and W_2 we divide the first equation by the second. We thus obtain

$$\frac{a}{b}=\frac{W_1 b}{W_2 a},$$

or

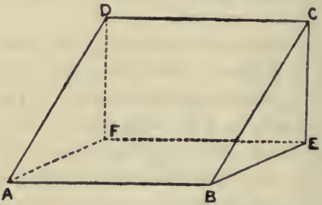
$$\left(\frac{a}{b}\right)^2=\frac{W_1}{W_2},$$

from which

$$\frac{a}{b}=\sqrt{\frac{W_1}{W_2}}.$$

124. Inclined Plane without Friction.

Let ABCD be a smooth plane inclined to the horizon ABEF; it is required to find what power P, acting in a specified direction in a plane perpendicular to AB, will support a given weight W on the inclined plane.



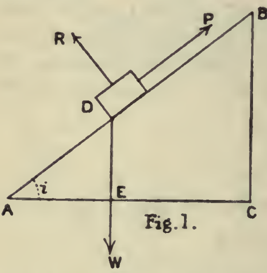


Fig. 1.

I. Let the power act along the inclined plane as in Fig. 1, which represents a section of the inclined plane perpendicular to the line of intersection of the inclined plane and the horizon. In this case the given body D is in equilibrium under the action of three forces: (1) the weight W vertically downwards, (2) the power P along the inclined plane, (3) the reaction R of the plane in a direction perpendicular to the plane. Since D is in equilibrium the sum of the components of all the forces along the plane must be zero. The component of W down the plane is W cos ADE, or W sin DAE, or W $\sin i$, if i be the inclination of the plane to the horizon. Hence

$$P-W\sin i=0, \text{ or } P=W\sin i.$$

Similarly the sum of the components perpendicular to the plane must be zero. The component of W perpendicular to the plane is $W \cos i$, and this must be equal to R. Hence

$$R = W \cos i,$$

which gives the reaction of the plane.

II. Let the power act parallel to the horizon as in Fig. 2. Resolving along and perpendicular to the plane, and expressing that the sums of the components along and perpendicular to the plane are zero, we get the two equations

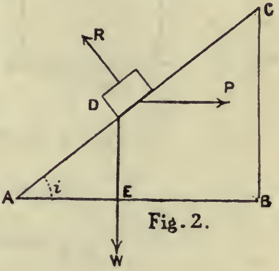


Fig. 2.

$$P \cos i - W \sin i = 0 \ldots\ldots(1)$$

$$P \sin i + W \cos i - R = 0 \ldots(2)$$

From equation (1) we get

$$P = W \frac{\sin i}{\cos i} = W \tan i \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

which gives P in terms of W.

From equation (2) we get

$$\begin{aligned} R &= P \sin i + W \cos i \\ &= W \tan i \sin i + W \cos i \\ &= W\left(\frac{\sin^2 i}{\cos i} + \cos i\right) \\ &= W \frac{\sin^2 i + \cos^2 i}{\cos i} \\ &= \frac{W}{\cos i}, \end{aligned}$$

which gives R in terms of W.

III. Let the power act in a direction making an angle a with the inclined plane, as in Fig. 3.

Expressing that the sums of the components along and perpendicular to the plane are zero, we get the two equations

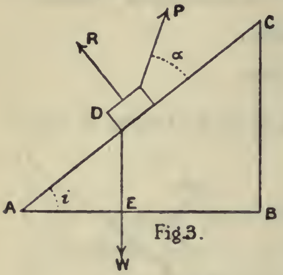


Fig. 3.

$$P\cos\alpha - W\sin i = 0 \quad \ldots\ldots(1)$$

$$R + P\sin\alpha - W\cos i = 0 \quad \ldots(2)$$

From equation (1) we find

$$P = \frac{W\sin i}{\cos\alpha} \quad \ldots\ldots\ldots\ldots(3)$$

From equation (2) we find

$$\begin{aligned} R &= W\cos i - P\sin\alpha \\ &= W\cos i - W\frac{\sin\alpha\sin i}{\cos\alpha} \\ &= W\frac{\cos i\cos\alpha - \sin\alpha\sin i}{\cos\alpha} \\ &= W\frac{\cos(i+\alpha)}{\cos\alpha}. \end{aligned}$$

Example.—An inclined plane rises 3 feet for every 5 feet measured along the plane. What force is required to support a weight of 100 lbs. (1) when the force acts horizontally, (2) when the force acts parallel to the plane?

If in Figs. 2 or 3 AC is 5 feet, then BC is 3 feet, and therefore $\sin BAC$ or $\sin i$ is $\frac{BC}{AC}$ or $\frac{3}{5}$. Also $\tan i$ is equal to $\frac{BC}{AB}$ or $\frac{3}{\sqrt{(25-16)}}$, that is, $\tan i$ is $\frac{3}{4}$.

(1) When the power acts horizontally

$$P = W\tan i.$$

In this case $$P = 100 \times \frac{3}{4} = 75 \text{ lbs.}$$

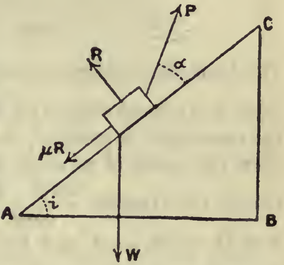
(2) When the power acts parallel to the plane

$$P = W\sin i.$$

In this case $$P = 100 \times \frac{3}{5} = 60 \text{ lbs.}$$

125. Inclined Plane with Friction.

A body D rests on a plane AC inclined to the horizon at an angle i, the coefficient of friction between the body and the plane being μ. A force P acts on the body in a direction inclined at an angle α to the inclined plane. It is required to find the relation between P and the weight W of the body when P is on the point of making the body slide up the plane.



When the body is on the point of sliding up the plane the forces acting on the body are the force P, the weight W vertically downwards, the reaction R of the plane, and the friction μR down the plane. When the body is just on the point of sliding these forces are in equilibrium. The sum of the components along and perpendicular to the plane must therefore be zero. Hence

$$P\cos\alpha - \mu R - W\sin i = 0. \qquad \text{(1)}$$

$$R + P\sin\alpha - W\cos i = 0 \qquad \text{(2)}$$

Multiplying equation (2) by μ and adding it to equation (1), we get

$$P(\cos\alpha + \mu\sin\alpha) - W(\sin i + \mu\cos i) = 0,$$

from which

$$P = \frac{W(\sin i + \mu\cos i)}{\cos\alpha + \mu\sin\alpha}.$$

Substituting this value of P in equation (2), the student will have no difficulty in proving that

$$R = \frac{W(\cos i\cos\alpha - \sin i\sin\alpha)}{\cos\alpha + \mu\sin\alpha} = \frac{W\cos(i+\alpha)}{\cos\alpha + \mu\sin\alpha}.$$

In order to find what is the least force which will be sufficient to drag the body up the plane we must find the value of α which will make the fraction

$$\frac{W(\sin i + \mu\cos i)}{\cos\alpha + \mu\sin\alpha}$$

least. This fraction will be least when the denominator is greatest. Now if ϕ be the angle of friction, $\tan\phi=\mu$, and therefore the denominator is equal to

$$\cos a+\tan\phi\sin a=\cos a+\frac{\sin a\sin\phi}{\cos\phi}$$
$$=\frac{\cos a\cos\phi+\sin a\sin\phi}{\cos\phi}=\frac{\cos(a-\phi)}{\cos\phi}.$$

The denominator therefore is greatest when $\frac{\cos(a-\phi)}{\cos\phi}$ is greatest. Since ϕ is a fixed angle, this last fraction will be greatest when the numerator is greatest, that is, when $\cos(a-\phi)$ is greatest. Now the cosine of an angle is greatest when the angle is zero. Hence the fraction $\frac{\cos(a-\phi)}{\cos\phi}$ is greatest when $a-\phi$ is zero, that is, when $a=\phi$, and its value is then $\frac{1}{\cos\phi}$. Hence the force necessary to drag a body up an inclined plane is least when the direction of the force is inclined to the inclined plane at an angle equal to the angle of friction. Also the value of the force in terms of W, i, and ϕ is by the above

$$\frac{\mathrm{W}(\sin i+\tan\phi\cos i)}{\dfrac{1}{\cos\phi}}$$
$$=\mathrm{W}(\sin i\cos\phi+\sin\phi\cos i)$$
$$=\mathrm{W}\sin(\phi+i).$$

By putting $i=0$ we get the case of a body being dragged along a horizontal plane. The least force necessary to drag the body along the plane is obtained by putting $i=0$ in the expression just found. Hence the least force is W $\sin\phi$, and it is inclined to the horizon at an angle ϕ.

EXAMPLE.—Find the least force which is necessary to drag a body up a plane inclined to the horizon at an angle of 30°, and compare it with the least force required to drag the body along a horizontal plane, the coefficient of friction in each case being

$$\frac{1}{\sqrt{3}}.$$

The angle whose tangent is $\frac{1}{\sqrt{3}}$ is the angle 30°; hence ϕ in this case is 30°. To find the least force necessary to drag a body up the inclined plane, we put $\phi=30°$, and $i=30°$ in the formula $W\times\sin(\phi+i)$. Hence if P_1 be the force

$$P_1=W\sin(30°+30°)=W\sin 60°$$
$$=W\frac{\sqrt{3}}{2}.$$

To find the least force necessary to drag the body along the horizontal plane, we put $i=0$ and $\phi=30°$. Hence if P_2 be the force

$$P_2=W\sin 30°$$
$$=\frac{W}{2}.$$

Therefore $$P_1 : P_2=W\frac{\sqrt{3}}{2} : \frac{W}{2}$$
$$=\sqrt{3} : 1.$$

Hence the least force necessary to drag the body up the inclined plane is $\sqrt{3}$ times the least force necessary to drag it along a horizontal plane.

126. The Screw.

In the ordinary form this machine is a screw passing through a fixed nut on the inner surface of which is cut a groove which fits accurately to the thread of the screw. The screw is actuated by a force, called the power, applied at the extremity of an arm fixed to the screw at one end. When the arm is turned by the power the screw passes through the nut and exerts at the other end a pressure which may be used to overcome a resistance, usually called the weight.

A familiar example of the use of the screw is seen in the ordinary copying press, where the weight is the resistance of a book to compression. In a blacksmith's vice the screw is used for bringing the two jaws of the vice together. In many locomotive engines the screw forms part of the arrangement for applying the brake to the wheels of the tender.

The **step** of a screw is the distance between the threads measured along the direction of the length of the screw. When the arm of the screw makes a complete revolution the screw moves through a distance equal to the step. The relation between the power P and the weight W depends on the length of the arm and on the step.

We shall find the relation between P and W by the principle of the conservation of energy. We suppose the power to act in a direction at right angles to the arm.

Let a be the step of the screw and b the length of the arm. In a complete revolution of the arm the power acts through a space equal to the circumference of a circle of radius b, that is, through distance $2\pi b$. Hence the work done on the screw by the power in one revolution of the arm is $2\pi b \mathrm{P}$. Since the screw moves through the length of a step while the arm is making one complete revolution, the work done by the screw in the same time is $\mathrm{W} a$. By the principle of conservation of energy the work done on the screw is equal to the work done by the screw. Hence

$$\mathrm{W} a = 2\pi b \mathrm{P},$$

from which $$\frac{\mathrm{W}}{\mathrm{P}} = \frac{2\pi b}{a}.$$

EXAMPLE.—It is required to produce a pressure of 800 lbs. by means of a screw whose thread makes 2 turns in 3 inches, the length of the arm of the screw being 2 feet. Determine the power required.

Here the step is $\frac{3}{2}$ inches or $\frac{1}{8}$ foot, and the arm is 2 feet. Hence

$$a = \tfrac{1}{8},\ b = 2,\ \mathrm{W} = 800.$$

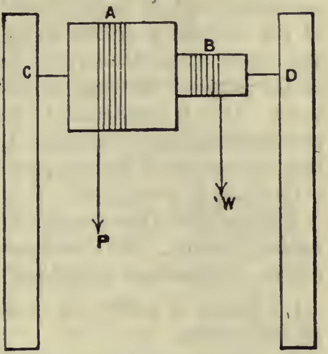
Substituting these values in the above equation, and taking $\pi = \frac{22}{7}$, we get

$$\frac{800}{\mathrm{P}} = \frac{2 \times \frac{22}{7} \times 2}{\frac{1}{8}},$$

from which $\mathrm{P} = 8$ lbs. very nearly.

127. The Wheel and Axle.

In this machine two cylinders A and B are fitted together so as to turn on the same axis CD. One cylinder A, which is called the wheel, has a much larger diameter than the other cylinder B, which is called the axle. Ropes are wrapped round the two cylinders in opposite directions so that when the machine is set in motion the end of one rope ascends and the end of the other descends. The power is applied at the extremity of the rope round the larger cylinder, and the weight at the extremity of the other rope. The power instead of being applied at the extremity of a rope passing over a wheel is sometimes applied by a handle as in the common windlass, or at the extremity of spokes as in the capstan.



Let W denote the weight fixed to the end of the rope wound round the axle; it is required to find what force P must act downwards at the end of the other rope in order to balance W. The machine is acted on by two forces only, the weight W and the power P, both acting vertically downwards. For equilibrium the moments of these forces round the axis must be zero. Let a and b denote the radii of the larger and smaller cylinder respectively; then

$$W\,b = P\,a,$$

from which $$\frac{W}{P} = \frac{a}{b}.$$

EXAMPLE.—The rope from the axle is attached to a ton weight placed on a smooth plane inclined to the horizon at an angle of 30°. Find the power necessary to keep this in equilibrium when the rope is parallel to the plane, the radii being 4 feet and 9 inches respectively.

The force acting parallel to the plane necessary to keep a ton weight from sliding down an inclined plane of 30° is, by Article 124,

$$20 \times \sin 30^\circ = 20 \times \tfrac{1}{2} = 10 \text{ cwt.}$$

This, therefore, is the tension of the rope attached to the axle, that is, $W = 10$ cwt. Also $a = 4$ feet, $b = 9$ inches $= \frac{3}{4}$ foot. Hence

$$\frac{10}{P} = \frac{4}{\frac{3}{4}},$$

from which $$P = \frac{15}{8} \text{ cwt.} = 1 \text{ cwt. } 98 \text{ lbs.}$$

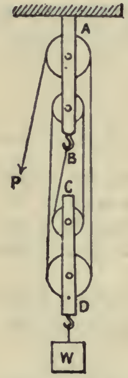
128. Pulleys.

A pulley is a thin wooden cylinder which can turn about an axis through its centre, there being cut on the circumference of the cylinder a groove along which a rope can run. We shall assume that when a rope passes over a pulley the tensions of the parts of the rope on either side of the pulley are the same. This is equivalent to neglecting friction and the imperfect flexibility of the rope. It follows from this assumption that the tension of the rope passing over any number of pulleys is the same in every part.

Pulleys are often combined together for the purpose of raising a weight. We shall consider the mechanical advantage of two combinations or systems of pulleys.

(1) *System of pulleys in which the same string passes round all the pulleys.*

In this system of pulleys there are two frameworks called blocks, of which the upper AB is fixed and the other CD is movable, each of the blocks containing any number of pulleys. One end of the string is attached to the lower part B of the fixed block, passes over the uppermost pulley of the lower block, then over the lowermost pulley of the upper block, then over the uppermost pulley but one of the lower block, and so on. From the lower end of the lower block is suspended a weight W, and it is required to find what force P acting at the free end of the string will be sufficient to support W.



In the figure there are two pulleys in each block, so that the lower block is suspended by four strings which may be considered without appreciable error to be all vertical. The tension of each of these strings is equal to P, so that W is supported by 4 tensions each equal to P. Hence

$$W = 4\,P.$$

In the same way if there are n pulleys in the lower block,

$$W = 2n\,P.$$

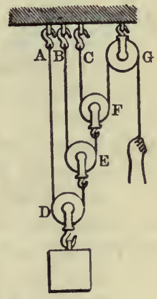
We can take the weight of the lower block and pulleys into account by supposing W to include that weight.

In the "block and tackle" in common use the pulleys in either block, instead of being placed one above another, all turn on the same horizontal axis.

(2) *System of pulleys in which a separate string goes round each pulley.*

This arrangement of pulleys is shown in the figure.

In this system of pulleys a string fixed at A passes under the lowermost pulley D, the other end being attached to the pulley E. Another string fixed at B passes under the pulley E, the other end being attached to the pulley F; and so on. The fixed pulley G is introduced for convenience and has no effect on the mechanical advantage of the arrangement. The weight W is hung from the lowermost pulley D.



The tension of the string passing under D is evidently ½ W, and this we may look upon as the weight suspended from E. It will follow that the tension of the string passing under E is $\frac{1}{2} \times \frac{1}{2} W = \frac{1}{2^2} W$. Similarly the tension of the string passing under F is $\frac{1}{2} \times \frac{1}{2^2} W = \frac{1}{2^3} W$, and this is the same as P. Hence

$$P = \frac{1}{2^3} W$$

$$\text{or} \qquad W = 2^3 . P.$$

Similarly if there are n movable pulleys,

$$W = 2^n P.$$

EXAMPLE.—Find what force would be necessary to support 5 cwt. by 3 movable pulleys, taking account that the weight of each pulley is 4 lbs.

The tension of the string passing under the lowermost pulley is

$$\tfrac{1}{2} (5 \text{ cwt.} + 4 \text{ lbs.}) = \tfrac{1}{2} \times 564 = 282 \text{ lbs.}$$

The tension of the string passing under the next pulley is

$$\tfrac{1}{2}(282+4)=143 \text{ lbs.}$$

The tension of the string passing under the third pulley is

$$\tfrac{1}{2}(143+4)=73\tfrac{1}{2} \text{ lbs.}$$

Hence the force required is a force of

73½ pounds weight.

EXAMPLES XXVIII.

1. Find the force required to drag a mass of 60 lbs. up a slope of 1 in 20, the force acting along the slope, and the coefficient of friction being ·5.

   What is the force required to drag the body down the slope?

2. A mass of 200 lbs. is drawn up a slope of 1 in 2 by a force of 180 lbs. weight, the direction of the force making an angle of 30° with the plane. What is the coefficient of friction between the mass and the plane?

3. Two inclined planes meet along a horizontal line 12 inches above their bases. The base of one is 9 inches and that of the other is 16 inches. A mass of 30 lbs. rests on the shorter plane, and is kept in equilibrium by a string which passes up the plane and over a smooth pulley at the top, and has attached to it a mass resting on the longer plane. Find the mass.

4. A uniform lever 4 feet long weighs 5 lbs. What weight at the end of the shorter arm will balance 12 lbs. at the end of the longer arm, the fulcrum being 1 foot from one end?

5. A body is weighed first in one pan and then in the other pan of a balance with unequal arms. The apparent weight of the body in the first determination is 29·62 grammes, and in the second it is 28·7 grammes. Find the true weight of the body and the ratio of the arms of the balance.

6. A rod AB whose length is 5 feet and weight 10 lbs. is found to balance itself if supported on a fulcrum 3 feet from one end A. If the rod were placed horizontally on two points, one under A and the other under B, what pressure would it exert on each point?

7. A lever AB of the first order, 8 feet long, has its fulcrum 2 feet from B. A weight of 5 lbs. is hung from A and one of 17 lbs. is hung from B. From what point must a weight of 2·5 lbs. be hung to keep the lever horizontal?

Where must the centre of gravity be if we are justified in leaving the weight of the lever out of account?

8. In the inclined plane if the angle between the power and the plane be equal to that between the plane and the horizon, and each be equal to the angle whose tangent is $\frac{1}{2}$, then will

$$2P = W.$$

9. In the inclined plane if R be the reaction of the plane when the power acts horizontally and R′ when it acts parallel to the plane, show that the weight is equal to $\sqrt{RR'}$.

10. A garden roller weighing 80 lbs. with its handle exactly counterpoised is drawn up a plank sloping at 30°, the handle being kept parallel to the plank. What force must be applied to the handle for this purpose, and what force will be necessary when the handle is turned into the horizontal direction?

11. Show that in the smooth inclined plane the work done in drawing a body up the inclined plane by a force parallel to the plane is equal to the work done in lifting the body vertically through the height of the plane.

12. Show that in the rough inclined plane the work done in drawing a body up the inclined plane by a force parallel to the plane is equal to the work done in dragging the body horizontally over the base of the plane, supposed equally rough, together with the work done in lifting the body vertically through the height of the plane.

13. The lever of a screw is 2 feet and the step of the thread is 2 inches. If pressure of 100 lbs. be applied to the lever, what pressure will be produced on the pressure board?

14. The step of the thread of a screw is $\frac{1}{8}$ of an inch, and the length of the lever is 6 inches. If a power of 15 lbs. acts at the end of the lever, find the pressure which will be exerted by the screw.

15. In the system of one fixed and one movable pulley, the power is 7 lbs., and the weight, including the movable pulley, is 11 lbs. Neglecting friction and the inertia of the pulleys find the tension of the cord and the accelerations of the power and the weight.

16. If the thread of a screw makes 25 turns in 3 inches and the arm is 24 inches, what force must be applied to sustain a weight of 1 cwt.?

17. A man standing in a scale attached to a pulley supports himself by holding down the free end of the rope. Find the pressure on the scale the strings being parallel.

18. In raising a weight of 500 lbs. by aid of a set of pulleys it is found that 30 feet of cord must be pulled in at the free end to raise the weight 5 feet. Show what power would be required and sketch the necessary arrangement.

19. A rod AB turning freely round a hinge at A rests at an inclination of 30° to the horizon with its end B against a smooth plane inclined at an angle of 60° to the horizon. To the other end B a thread is fastened which, passing over a smooth pulley at the top of the plane, hangs freely and has attached to it a weight equal to half that of the rod. Find the pressure on the inclined plane.

CHAPTER XII.—HYDROSTATICS.

FLUID PRESSURE.

129. Definitions.

In Article 3 we referred to the three states in which matter can exist—the solid, the liquid, and the gaseous state—and we stated that solids differ from liquids and gases in having a definite shape. Liquids and gases are classed together under the name of *fluids.*

Hydrostatics is the Science of the Equilibrium of Fluids.

Many fluids with which we are practically acquainted offer very little resistance to change of shape, and we are led to conceive of an ideal fluid in which there will be absolutely no resistance to change of shape. Such a fluid is called a **perfect fluid.** Thus—

A perfect fluid is one in which there is absolutely no resistance to change of shape.

Water and alcohol are approximations to a perfect fluid. On the other hand many fluids, such as oil, tar, treacle, jelly, &c.,

offer more or less resistance to change of shape. Such fluids are called **viscous fluids.** It is found by experiment that the resistance a viscous fluid offers to change of shape depends on the speed of the change, and becomes infinitesimal when the change is produced very slowly. Thus a viscous fluid in any circumstances will, if sufficient time be allowed, take up the form which a perfect fluid would take up immediately.

The difference between a perfect fluid and a viscous fluid can be expressed in another way. In a perfect fluid there is no frictional resistance to the sliding of one particle of the fluid over another particle, while in viscous fluids there is such a resistance. This difference is illustrated by pouring on the ground a quantity of a fluid such as water and of a very viscous fluid such as treacle. Between the particles of the water there is very little frictional resistance, so that the water spreads out immediately into a very thin sheet. The treacle also spreads out into the same form, but only after the lapse of considerable time.

From the above definition of a perfect fluid we infer that the pressure of such a fluid on any surface with which it is in contact is perpendicular to the surface. In viscous fluids there is frictional resistance in the direction of the tangent to the surface between the fluid and the surface. This frictional resistance becomes infinitesimal when the motion of the fluid over the surface becomes infinitely slow, and it is therefore to be neglected altogether for fluids at rest. In considering therefore the equilibrium of fluids it is not necessary to consider viscosity at all. We have therefore the following as the fundamental law of hydrostatics:—

The pressure of any fluid at rest on any surface with which it is in contact is perpendicular to that surface.

130. Compressibility of Liquids.

We stated in Article 3 that liquids maintain a definite size. This, however, is not strictly true. All liquids are more or less compressible. It is found by experiment that liquids, though they offer a great resistance to compression, have their volumes sensibly diminished by the application of a great pressure.

Thus water is compressed one-twenty-thousandth part of its volume by the application of a pressure of 14 pounds weight on every square inch.

Since the compressibility of liquids is so small it is usual in hydrostatics to neglect it altogether, and to consider liquids as incompressible fluids. The results obtained by the principles of hydrostatics on this supposition do not differ materially from the results obtained by experiment and observation.

131. Pressure at a Point of a Fluid.

In measuring the pressure of a fluid on a surface with which it is in contact we have first to select some force as our unit of force, and some area as our unit of area.

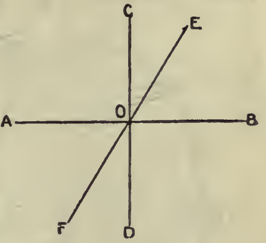
If the pressure of the fluid over the surface is constant, the measure of the pressure of the fluid on the surface is the number of units of force on every unit of area. Thus the pressure of steam on the surface of a boiler is measured by pressure on every square inch, the pressure being measured in pounds weight.

If the pressure of a fluid on a surface with which it is in contact is not constant, the pressure *at* a given point of the surface must be measured as all varying quantities are measured. The pressure at the given point of the surface is the pressure which would be exerted on unit of area if the pressure on that unit of area were the same as at the given point. This conception will offer no difficulty to the student, as he has already met with it in the measure of a varying velocity and a varying acceleration.

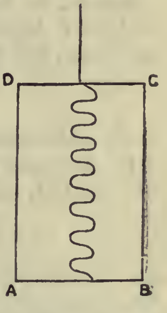
The pressure *at* a point in the interior of a fluid is the pressure which would be exerted on unit of area placed at that point. Thus the pressure at a point a mile below the surface of the sea is a pressure of about 2270 lbs. weight on every square inch.

132. Equality of Pressure in all Directions.

At a point in the interior of a fluid the pressure is the same in all directions. Thus if O be a point in the interior of a fluid, the pressure (supposed to be uniform) on the unit of area AB which is horizontal is equal to the pressure (supposed to be uniform) on the unit of area CD which is vertical, and generally is equal to the pressure on any unit of area EF. This is proved experimentally by the following experiment:—



Let ABCD be a cylinder, closed at one end AB, and fitted at the other end with a piston CD which is connected with the end AB by a spiral spring. If this cylinder be placed so that the centre of the piston is at a given point in the interior of a fluid, the piston will be pressed inwards to a point at which the fluid pressure on the piston is equal to the force exerted by the compressed spring. Whatever be the position of the cylinder, whether vertical, horizontal, or inclined at any angle to the vertical, as long as the centre of the piston coincides with the given point of the fluid, it will be found that the piston will always be forced inwards through the same distance. This clearly shows that, whatever be the position of the piston, the pressure of the fluid on the piston is always the same as long as the centre of the piston coincides with the given point of the fluid.

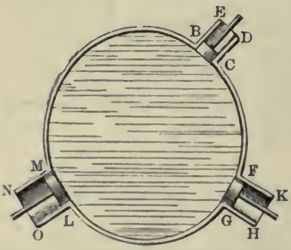


Hence the pressure at a given point of a fluid must be the same in all directions.

133. Transmission of Fluid Pressure.

Any pressure communicated to any part of a fluid is transmitted equally in all directions. This is proved experimentally by the following experiment:—

Let the figure represent a closed vessel with any fluid compressible or incompressible, and let there be inserted into the vessel cylinders BCDE, FGHK, LMNO, fitted respectively with pistons BC, FG, LM.



The pressure of the fluid on the pistons will tend to press the pistons outwards. Let these pressures be balanced by weights or some other means. Then if one piston, the piston BC suppose, is pressed inwards with an additional pressure, this additional pressure will be communicated to the pistons FG and LM.

Let A_1, A_2, A_3 denote the areas of the pistons BC, FG, LM, and let P denote the additional pressure exerted inwards on the piston BC, then

$$\frac{P}{A_1}$$

is the additional pressure on every unit of area of the piston AB. If Q and R denote the additional outward pressures on FG and LM, then

$$\frac{Q}{A_2} \quad \text{and} \quad \frac{R}{A_3}$$

are the additional outward pressures on each unit of area of the pistons FG and LM. Now it is found by experiment that

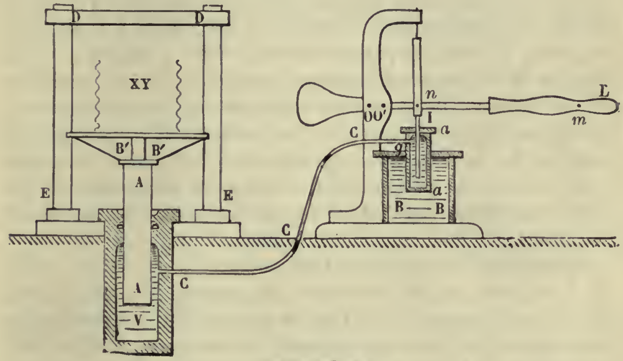
$$\frac{P}{A_1}=\frac{Q}{A_2}=\frac{R}{A_3};$$

that is, the additional pressures per unit of area on the pistons FG and LM are each equal to the additional pressure per unit of area on the piston BC.

134. Bramah's Press.

The action of Bramah's hydraulic press depends on the property of the equal transmission of pressure by water.

The machine consists of two parts. In one part there is a chamber BB containing water, into which is fitted a cylinder *aa*. In this cylinder there works a rod of metal I, called a plunger, which is actuated by power applied at the extremity *m* of a handle L, movable about the fixed end OO′ and attached to the plunger at the point *n*. At the lower end of the cylinder there is a valve opening upwards, and at a point *g* in the side of the cylinder is placed another valve opening outwards from the cylinder into a pipe CC communicating with the other part of the machine.



Bramah's Press.

This connecting pipe opens into a large cylinder V in which there works a large piston AA. On the upper end of this piston is fixed a plate B′B′, which moves in the framework DD, EE. In using the machine as a press, the body, XY, which is to be compressed is placed on the movable plate B′B′, and, when the machine is worked, is pressed between B′B′ and DD, the upper end of the framework.

In working the machine the end *m* of the lever L is first raised, and the plunger I is thereby raised in the cylinder *aa*. The pressure of water on the lower end of the cylinder opens the valve at that point, and water enters the cylinder from the chamber BB. The end *m* of the lever L is then pressed down, and this pressure is communicated to the water in the cylinder, closing the valve at the bottom of the cylinder and opening the valve at the point *g*. Thus on the downward stroke of the plunger the water in the cylinder *aa* will be forced through the valve *g* and the pipe CC into the large cylinder V. When the lever is raised again the water in the cylinder V will be prevented from returning to the cylinder *aa* by the closing of the valve at *g*.

At every stroke of the lever water will be forced from the cylinder *aa* into the larger cylinder V until the chamber V is full of water. When this stage has been arrived at, the pressure exerted by the plunger I on the water in the cylinder *aa* will be communicated equally to the piston AA, so that the piston AA will be pressed upwards with a pressure which bears to the downward pressure exerted by the plunger the ratio of the area of the section of the piston AA to the area of the section of the plunger I. The mechanical advantage can therefore be increased to any extent by increasing the ratio of the area of the piston AA to the area of the section of the plunger I. Mechanical advantage is also gained by the use of the lever L.

To calculate the mechanical advantage.

Let the area of the section of the piston AA be p times that of the plunger I; let Om be denoted by a, m being the point where the power is supposed to be applied, and let On be denoted by b. If a power P is exerted on the lever at m, the pressure exerted downwards by the plunger will, by the principle of the lever, be

$$\frac{\mathrm{P}a}{b}.$$

By the principle of the equal transmission of pressure the pressure is transmitted equally to all parts of the lower end of the piston AA′. Hence, since the area of the section of the piston AA is p times that of the plunger I, it follows that the piston AA will be pressed upwards with a force

$$\frac{p\mathrm{P}a}{b}.$$

This therefore is the force which is exerted on the body XY between the movable plate and the upper end of the framework.

EXAMPLE.—$b=1$ foot, $a=5$ feet, area of section of the cylinder $I=\frac{1}{2}$ sq. foot, area of section of large cylinder AA $=30$ sq. feet; then

$$p=\frac{30}{\frac{1}{2}}=60, \quad \text{and} \quad \frac{a}{b}=\frac{5}{1},$$

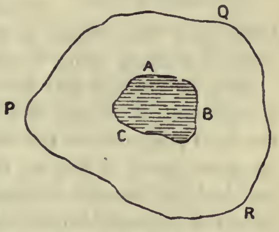
so that the mechanical advantage is

$$60\times 5=300.$$

A pressure of 10 pounds weight exerted at the end B of the lever will cause the piston AA to exert a pressure of 3000 pounds weight on the body XY.

135. In any fluid at rest any portion of the fluid may be supposed to become solid without in any way disturbing the equilibrium of that portion of the fluid. By means of this supposition we are able to apply the principles of statics to fluids.

Thus if PQR be a fluid, in considering the equilibrium of any portion ABC of the fluid, we may suppose that portion to become solid. The fluid pressures exerted by the surrounding fluid, as well as the forces, such as gravity, which may act on ABC will all be unchanged, so that the equilibrium of ABC will not be disturbed.



136. The Action of Gravity on Fluids.

Every particle of a fluid is acted on by gravity, and therefore has weight.

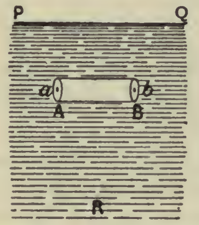
In this chapter we shall suppose the fluid to be incompressible and to be of uniform density throughout.

The pressure will be the same in all directions at a given point, but will vary from point to point.

We proceed to prove three important propositions regarding the pressures at different points of an incompressible fluid of uniform density acted on by gravity.

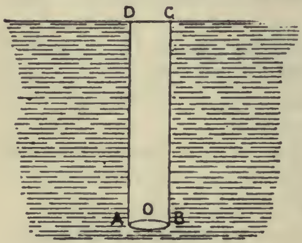
I. *The pressure is the same at every point of a horizontal plane.*

Let PQR be a mass of fluid of uniform density throughout. Let a very small horizontal column AB of liquid of circular section be supposed to become solid. This will not disturb the equilibrium, and the column will still remain at rest under the action of the same forces as before. These forces are (1) the weight of the column vertically downwards, (2) the pressures of the surrounding fluid on the surface of the column, (3) the pressures of the fluid on the end sections A and B along the axis of the column. The first two forces acting at right angles to the axis of the column have no component along that axis. It follows, since the column is at rest, that the pressures on the two end sections along the axis must be equal.



Now the column may be taken so small that the pressure at every point of an end section will be the same. Since then the total pressures on the end sections are equal, and since the areas are equal, it follows that the pressures at the centres *a* and *b* of the two end sections are equal. But *a* and *b* are two points in the same horizontal line; hence the proposition.

II. *The pressure at any point within an incompressible fluid is equal to the weight of a column of the fluid whose section is unit area and whose height is equal to the depth of the point below the surface of the fluid.*



Let O be any point of a fluid, and let a surface AB of unit area be placed horizontally at O. The pressure over AB is constant, so that the whole pressure on AB is the measure of the pressure at O. Imagine the column of fluid ABCD of which AB is the base to become solid. This column of liquid is under (1) the pressure of the surrounding fluid on the column, (2) the weight of the column, (3) the pressure upwards on the base AB. Since the pressures of the surrounding fluid are perpendicular to the column, they act horizontally and therefore have no vertical component. Hence the two remaining forces which act vertically must balance each other; that is, the weight of the column ABCD is equal to the pressure on the base AB.

From this it follows that the pressure at any point of a fluid varies as the depth of the point below the surface.

Hence also if the pressures at two points are equal these two points must be at equal distances below the surface of the fluid.

III. *The surface of an incompressible fluid at rest is horizontal.*

This follows immediately from the two preceding propositions. For any two points on the surface may be considered as at equal depths below the surface, and therefore the pressures at these two points must be equal. Hence by Prop. I. the two points must be in the horizontal plane.

Corollary.—In the case of a fluid acted on by no forces, the pressure is the same at every point. For by Prop. II. the pressure is the same at all depths, since the weight of the vertical column ABCD is zero. Also, as in Prop. I., the pressure is the same at every point of a horizontal plane. Hence the pressure is the same at every point.

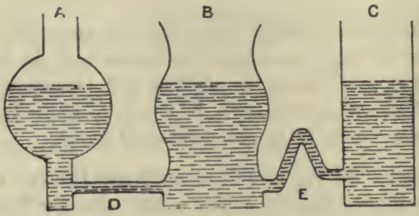
137. In the preceding Article we have neglected the presence of the atmosphere. We shall afterwards see how the atmosphere increases the pressure at a point in the interior of a fluid.

We have just proved that the surface of an incompressible fluid is horizontal. This we have proved by assuming that gravity acts in parallel lines, or in other words that vertical lines are parallel. This, as we have seen in Article 68, is not exactly true. Vertical lines converge to the centre of the earth, and the surface of a fluid at rest is really part of a spherical surface. Thus, though there is no practical error in taking the surface of a small lake as horizontal, the fact that the surface of the ocean is spherical has to be taken into account in practical problems in navigation.

The form of the surface of a large sheet of water is also affected by variations in the pressure of the atmosphere.

138. The proposition that the surface of water at rest is horizontal is often stated practically under the form that "water maintains its level." Thus if there are several vessels in communication containing water, the surfaces of the water in the different vessels will all lie in the same horizontal plane.

Thus if there be three vessels A, B, and C, A and B being connected by a pipe D, and B and C by a pipe E, and if water be poured into the vessel A, the water will rise to the same level in the three vessels. This is true even though the connecting pipe E is not horizontal, if the highest point of the pipe is below the level of the water in B.



It is on this principle that water from a reservoir is distributed by pipes throughout a town. The pipes need not be laid horizontally. The only precaution that must be taken is, that they shall not be laid over a hill the highest point of which is higher than the level of water in the reservoir.

139. Whole Pressure on a Horizontal Surface.

If a surface is immersed horizontally in an incompressible fluid of uniform density, the pressure at every point of the surface is the same, and therefore the whole pressure on the surface will be found by multiplying the number of units of area in the surface by the pressure on unit of area.

Let d be the depth of the surface below the surface of the fluid, A the number of units of area in the surface, and w the weight of unit volume of the fluid.

Since the volume of the column on unit area placed horizontally at a depth d is d units of volume, the weight of the column is wd, and this therefore is the pressure on unit of area. The area of the surface is A units of area, and therefore the whole pressure on the surface is

$$\mathrm{A}dw.$$

The weight of a cubic foot of water may for all practical purposes be taken to be 62·5 pounds weight, so that the pressure on a horizontal surface of A square feet at a depth of d feet below the surface of water is

$$62{\cdot}5\ \mathrm{A}d \text{ pounds weight.}$$

The weight of a cubic centimetre of water is 1 gramme weight, so that the pressure on a horizontal surface of A square centimetres at a depth of d centimetres is

$$\mathrm{A}d \text{ grammes weight.}$$

EXAMPLE.—A cube, each of whose edges is 2 feet long, stands on one face on the bottom of a vessel containing water 4 feet deep. Find the total pressure on the upper face of the cube.

In this case the upper face of the cube is 2 feet below the surface of the water, so that $d=2$. The area of the face of the cube is 4 feet, so that $\mathrm{A}=4$. Hence the whole pressure on the face of the cube is

$$=62{\cdot}5\times4\times2=500 \text{ pounds weight.}$$

140. Whole Pressure on a Surface not Horizontal.

When the surface is not horizontal the pressure will vary from point to point of the surface. To find the whole pressure in this case we take the average pressure over the surface and multiply that by the number of units of area in the surface.

It can be proved that the average pressure over any surface immersed in a fluid is equal to the pressure at the C.G. of the surface. We do not give the general proof of this, but the student will easily see that it is true for any parallelogram immersed in a fluid with two edges parallel to the surface of the fluid. The pressure varies as the depth, and therefore the average pressure on the parallelogram will be half the sum of the pressures at the upper and lower edges, or, what is the same thing, will be equal to the pressure at a point whose depth is midway between the depths of the upper and lower edges. Now the depth of the C.G. of the parallelogram is half the sum of the depths of the upper and lower edges. Hence the proposition is true for the case of a parallelogram two of whose edges are parallel to the surface of the fluid.

Assuming the truth of the proposition in general, we have the following rule for finding the whole pressure of a surface immersed in any position in a fluid:—

Multiply together the area of the surface, the depth of the C.G. of the surface, and the weight of unit volume of the fluid.

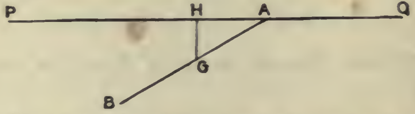
Example 1.—Find the pressure on one of the upright faces of the cube referred to in the example of the preceding Article.

The depth of the C.G. of an upright face is evidently $\frac{1}{2}(2+4)$ or 3 feet, so that the whole pressure on that face is

$$=62{\cdot}5\times4\times3=750 \text{ pounds weight.}$$

Example 2.—A square board whose edge is 2 feet long is immersed in water with one edge just in the surface. The board is inclined to the horizon at an angle of 30°. Find the pressure on one side of it.

Let the figure represent a section through the C.G. of the board, perpendicular to the plane of the board, PQ being the line in which the surface of the fluid is cut by the cutting plane, and AB the line in which the board is cut.



G, the C.G. of the board, is the middle point of AB; and if GH be drawn perpendicular to PQ, GH is the depth of the C.G. of the board. Now

$$GH=AG\sin 30^\circ=\tfrac{1}{2}AG=\tfrac{1}{4}AB$$
$$=\tfrac{1}{4}\times2\text{ feet}=\tfrac{1}{2}\text{ foot.}$$

Hence the whole pressure on one side of the board

$$=\tfrac{1}{2}\times4\times62{\cdot}5=125 \text{ pounds weight.}$$

141. Centre of Pressure.

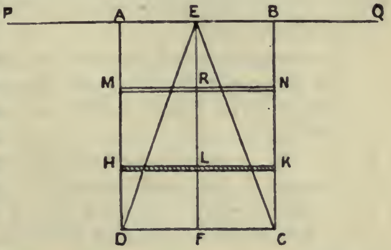
If a surface is immersed in a fluid, the point at which the resultant of all the fluid pressures on the surface acts is called the **centre of pressure** of the surface.

142. Centre of Pressure of a Rectangle Immersed in a Fluid with one Edge in the Surface.

In general the determination of the centre of pressure requires the use of the integral calculus. In one or two cases, however, the position of the centre of pressure can be determined by elementary considerations.

Let ABCD be a rectangle immersed vertically in a fluid with one edge AB in the surface PQ of the fluid.

Let the area of the rectangle be divided into an infinite number of very thin strips, all parallel to AB. Let HK be one of these strips. Let EF be the vertical straight line bisecting the rectangle.



Since the strip HK is infinitely thin, the pressure over that strip will be uniform, and the resultant will be a pressure acting at right angles to plane of rectangle at the point L, where EF cuts HK. Similarly, the resultant pressure on the strip MN will be a single force acting at right angles to plane of rectangle at R where EF cuts MN. Now the strips may be so taken that the area of each strip is the same, and therefore, since pressure varies as depth,

resultant pressure on HK : resultant pressure on MN = EL : ER.

It will follow that the resultant pressure on the whole rectangle will be the resultant of a system of parallel forces acting at points along EF, the magnitude of each force being proportional to the distance of the point from the point E.

Now by referring to Article 70 the student will easily see that the problem of finding where the resultant of this system of parallel forces acts reduces to that of finding at what point of EF the C.G. of the triangle EDC lies. We know that it lies at a distance $\frac{2}{3}$ EF from E. We infer that the centre of pressure of the rectangle ABCD lies in EF at a distance $\frac{2}{3}$ EF below the surface AB.

The above proof will still obtain if the rectangle be rotated about the edge AB into any position inclined to the vertical, since the pressures on the strips HK and MN will still be proportional to EL and ER.

EXAMPLES XXIX.

[In the following Examples the weight of a cubic foot of water is taken to be 62·5 lbs. weight, and the weight of a cubic centimetre to be 1 gramme weight.]

1. A rectangular board, a foot square, is immersed in water with its upper edge 10 feet below the surface and horizontal, the surface being vertical. Find the total pressure on one side of the board.

2. Find the pressure on each side of a rectangular board 14 feet long and 6 feet broad immersed in water with its length horizontal, and with its upper edge 5 feet and its lower edge 9 feet below the surface of the water.

3. Find the pressures on the bottom and on one of the sides of a cubical vessel 3 metres deep and filled with water.

4. The pressure of water on an embankment 100 feet long is 253,125 lbs. Required the depth of the water.

5. A canal lock is 12 feet wide, 8 feet deep, and 50 feet long. Find the total pressure on the gate and on the bottom of the lock.

6. A swimming bath contains water of the uniform depth of 4 feet. The sides are vertical and 50 feet long. Find in pounds weight the pressure on one side.

7. A cistern is full of water. Compare the pressure on the upper half of one side with the pressure on the lower half.

8. A hollow cylinder full of water, and with both ends closed, stands with its axis first vertical and then horizontal. Show that the pressure on the side of the cylinder in the first case is to the pressure in the second case as height is to diameter.

9. A triangular area is immersed in a liquid with the base horizontal. The vertex is at a depth of 12 feet and its base of 9 feet. The area of the triangle is 6 square feet. If a cubic foot of the liquid weighs 70 lbs., find the total pressure on the triangle.

10. Two equal small areas A and B are marked on a side of a reservoir full of water at different depths below the surface of the water. The pressure on A is 4 times the pressure on B, but if the surface of the water in the reservoir were lowered a foot the pressure on A would be five times the pressure on B. At what depth were A and B below the surface in the first instance?

11. A rectangular hole ABCD, whose lower side CD is horizontal, is made in a side of a reservoir, and is closed by a door whose plane is vertical. The door can turn freely round a hinge coinciding with CD. Calculate the force that must be applied to AB to keep the door shut, taking AB as 1 foot and AD 12 feet long, and that the water rises to the level of AB.

12. A cylindrical vessel open at the top, and containing water, is gradually tilted, none of the water being spilt, till the pressure on the base is diminished by a half. What is the inclination of the axis of the cylinder to the vertical?

The question plainly presupposes that the whole of the base of the cylinder remains under water. What, in terms of the radius of the base, is the smallest height of the cylinder and the smallest quantity of water for which this condition would be fulfilled?

13. A rectangular reservoir is full of water, its length and depth being respectively 10 feet and 5 feet. A diagonal line is drawn along one of its vertical sides, dividing it into two triangles. Find the magnitude of the resultant of the fluid pressures on each of these triangles. Why is the answer independent of the width of the reservoir?

14. A triangle ABC whose plane is vertical is wholly immersed in water in such a manner that the angle A is in the surface, and the bisector of the angle B is horizontal, and meets AC in D. Show that if $AB = \frac{3}{5}$ BC, the whole pressure on the triangle ABD is to the whole pressure on the triangle BCD as 9 is to 35.

15. A vessel with vertical sides has a horizontal base in the form of a regular hexagon. If each side of this hexagon is 1 inch, show to what height the vessel must be filled with a liquid in order that the total pressure on each vertical side may be equal to that on the base.

16. Find the total pressure in grammes weight on a triangle of area 10 square centimetres whose corners are at depths 3, 4 and 5 centimetres in water.

17. A triangle has its base horizontal and its vertex in the surface of a liquid. Divide it by lines parallel to the base into three parts on which the pressures are equal.

18. A cylinder, filled with liquid, with its axis vertical, is divided by horizontal sections into 5 annuli, so that the pressure on each is equal to the pressure on the base. If the radius of the base is 10 inches, determine the height of the cylinder and the breadth of the third annulus.

19. A rectangular area ABCD is just immersed in water with the side AB in the surface. Find a point P in AB such that the pressure on the triangle APD may be one-fifth of the whole pressure on the rectangle.

CHAPTER XIII.

DENSITY AND SPECIFIC GRAVITY.

143. Density.

In Article 29 we defined the density of a body to be the mass of unit volume of the body.

The density of a body depends on the temperature of the body, the general law connecting temperature and density being that the density decreases as the temperature increases. An exception to this law is found in the case of water. It is found by experiment that water has its greatest density at the temperature of 4° on the Centigrade scale of temperature, or, what is the same thing, at the temperature of 39°·2 on the Fahrenheit scale of temperature. This temperature is therefore referred to as the maximum density temperature of water. If the density of water at that temperature be taken as unity, the density at the temperature 0° on the Centigrade scale, that is, at the freezing point, is ·9999, and the density at 100° on the Centigrade scale, that is, at the boiling point, is ·9587.

It is also found by experiment that the mass of a cubic centimetre of water at the maximum density point is 1·000013 grammes. As this exceeds one gramme by an extremely small amount, there is no appreciable error in ordinary calculations in taking the mass of a cubic centimetre of water as exactly equal to the mass of one gramme. Hereafter we shall always take the mass of one cubic centimetre of water to be one gramme.

If then we take the mass of one gramme as the unit of mass, and the volume of one cubic centimetre as the unit of volume, the density of water at the maximum density point of water will be denoted by the number 1, and the density of any other body will be the number of grammes in a cubic centimetre of the body.

The term density is sometimes defined not absolutely as above but relatively, as denoting the mass of any volume of a body to the mass of the same volume of water at the maximum density point. The two definitions will lead to the same result if in measuring the absolute density the cubic centimetre is taken as the unit of volume and the gramme as the unit of mass.

Thus when we say that the density of sea water is 1·027, we may be taken to mean, either that the mass of any volume of sea water (say a cubic foot) is 1·027 times the mass of a cubic foot of pure water, or that the mass of a cubic centimetre of sea water is 1·027 grammes.

EXAMPLE 1.—What is the capacity in cubic centimetres of a vessel which holds 81576 grammes of mercury at a certain temperature, the density of mercury at that temperature being 13·596?

Since the density of mercury is 13·596, the mass of a cubic centimetre of mercury is 13·596 grammes, and since the vessel contains 81576 grammes of mercury, the capacity of the vessel

$$=\frac{81576}{13{\cdot}596}=6000 \text{ cubic centimetres.}$$

Example 2.—Find the mass of 100 feet of a copper wire of density 8, the diameter of the section of the wire being one centimetre.

The area of the section of the wire

$$=\pi\left(\frac{1}{2}\right)^2=\frac{22}{7}\times\frac{1}{4}=\frac{11}{14} \text{ square centimetres;}$$

and the volume of the wire

$$=\frac{11}{14}\times 100=\frac{550}{7} \text{ cubic centimetres.}$$

Hence the mass of the wire

$$=\frac{550}{7}\times 8=\frac{4400}{7}=628\tfrac{4}{7} \text{ grammes.}$$

144. Specific Gravity.

The specific gravity of a body is the ratio of the weight of any volume of the body to the weight of the same volume of water.

Since masses are proportional to their weights at the same place on the earth's surface, it follows that the number measuring the specific gravity of a body will be the same as the number measuring the density of the body defined relatively to water.

The student must however carefully notice that the ideas involved in the terms density and specific gravity are entirely different. The definition of specific gravity introduces the force of gravity, whereas the definition of density does not involve the idea of force at all. The physical connection between density and specific gravity is the same as that between the mass of a body and the weight of a body.

Example 1.—The specific gravity of petroleum is ·88; a quart of of water weighs 40 oz. Find how many gallons of petroleum will weigh $38\frac{1}{2}$ lbs.

A quart of petroleum will weigh $\cdot 88\times 40$ oz., and a gallon will weigh $\frac{\cdot 88\times 40\times 4}{16}$ lbs. $=8\cdot 8$ lbs. Hence the number of gallons of petroleum which will weigh $38\frac{1}{2}$ lbs. is

$$\frac{38\frac{1}{2}}{8\frac{4}{5}}=\frac{385}{88}=4\tfrac{3}{8} \text{ gallons.}$$

EXAMPLE 2.—A rod of uniform cross section 18 inches long weighs 3 oz., the specific gravity being 8·8. What fraction of a square inch is its cross section, the weight of a cubic inch of water being taken as 252 grains?

Let A denote the area of the cross section; then 18 A is the volume of the rod in cubic inches. Now the weight of a cubic inch of water is 252 grains, and therefore the weight of the rod is $8\cdot8 \times 18\,A \times 252$ grains. This we are told is 3 oz. or $3 \times 437\frac{1}{2}$ grains;

hence $$8\cdot8 \times 18\,A \times 252 = 3 \times 437\tfrac{1}{2},$$

from which $$A = \frac{625}{19008} \text{ inch.}$$

145. If w denote the weight of some volume of water taken as the unit volume, then the weight W of a volume V of a substance of specific gravity s is

$$W = wVs.$$

For the weight of V units of volume of water would be wV, and the weight of the same volume of the substance is found by multiplying the weight of an equal volume of water by the specific gravity. Hence the weight of the substance is wVs.

If a cubic foot be taken as the unit of volume, w is 62·5, and the weight of V cubic feet of a substance whose specific gravity is s, is $62\cdot5\,Vs$.

If a cubic centimetre be taken as the unit of volume, w is 1 gramme, and therefore the weight of V cubic centimetres of a substance whose specific gravity is s, is Vs.

146. Specific Gravity of a Mixture.

(1) Let a mixture be made of volumes v_1, v_2, v_3...... of substances whose specific gravities are s_1, s_2, s_3...... respectively: it is required to find the specific gravity of the mixture.

Let w denote the weight of unit volume of water; then it is evident from the preceding Article that the weights of the substances are

$$wv_1s_1,\ wv_2s_2,\ wv_3s_3.........,$$

and that the weight of the mixture is

$$w(v_1s_1 + v_2s_2 + v_3s_3 + ...).$$

The volume of the mixture is $v_1+v_2+v_3+\dots\dots\dots$, and therefore if s denote the specific gravity of the mixture,

$$ws\,(v_1+v_2+v_3+\dots\dots\dots)=w\,(v_1s_1+v_2s_2+v_3s_3+\dots\dots\dots),$$

from which $$s=\frac{v_1s_1+v_2s_2+v_3s_3+\dots\dots\dots}{v_1+v_2+v_3+\dots\dots\dots}$$

(2) Let a mixture be made of weights $w_1, w_2, w_3\dots\dots$ of substances whose specific gravities are $s_1, s_2, s_3\dots\dots$: it is required to find the specific gravity of the mixture.

The volumes of the substances are evidently

$$\frac{w_1}{ws_1},\ \frac{w_2}{ws_2},\ \frac{w_3}{ws_3}\ \dots\dots\dots,$$

and the volume of the mixture is

$$\frac{1}{w}\left(\frac{w_1}{s_1}+\frac{w_2}{s_2}+\frac{w_3}{s_3}+\dots\dots\dots\right).$$

The weight of the mixture is

$$w_1+w_2+w_3+\dots\dots\dots$$

Hence if s denote the specific gravity of the mixture

$$w_1+w_2+w_3+\dots\dots\dots=w\times\frac{1}{w}\left(\frac{w_1}{s_1}+\frac{w_2}{s_2}+\frac{w_3}{s_3}+\dots\dots\dots\right)s,$$

from which $$\frac{w_1+w_2+w_3+\dots\dots\dots}{s}=\frac{w_1}{s_1}+\frac{w_2}{s_2}+\frac{w_3}{s_3}+\dots\dots\dots$$

EXAMPLE 1.—3 parts by volume of a liquid of specific gravity ·8 are mixed with 7 parts of water, and the mixture shrinks in the ratio of 25 to 24. Find the specific gravity of the mixture.

Here $v_1=3$, $v_2=7$, $s_1=\cdot 8$, $s_2=1$, and the total volume of the mixture is $\frac{24}{25}\times 10$; hence if s denote the specific gravity of the mixture,

$$\begin{aligned}\frac{24}{25}\times 10\times s&=v_1s_1+v_2s_2\\&=3\times\tfrac{4}{5}+7\times 1,\end{aligned}$$

from which $$s=\frac{47}{48}.$$

EXAMPLE 2.—The specific gravity of gold being 19·26, and that of silver 10·47, calculate the amount of gold in an alloy of these metals whose specific gravity is 16 and weight 650 grains.

Let w_1, w_2 be the weights, and s_1, s_2 the specific gravities, of gold and silver respectively; then $s_1 = 19{\cdot}26$, $s_2 = 10{\cdot}47$, and

$$w_1 + w_2 = 650 \quad \ldots\ldots\ldots\ldots (1)$$

Also, since the specific gravity of the alloy is 16,

$$\frac{w_1}{s_1} + \frac{w_2}{s_2} = \frac{650}{16},$$

or

$$\frac{w_1}{19{\cdot}26} + \frac{w_2}{10{\cdot}47} = \frac{650}{16} \quad \ldots\ldots\ldots\ldots (2)$$

Solving equations (1) and (2) for the two unknowns w_1 and w_2, we get

w_1, the weight of gold $= 492{\cdot}25$ grains,
w_2, the weight of silver $= 157{\cdot}75$ „

147. Pressure on a Surface Immersed in a Fluid of given Specific Gravity.

We saw in Article 139 that the pressure on a surface of area A immersed in a fluid, the weight of unit volume of which is w, is

$$w\mathrm{A}d,$$

where d denotes the depth of the C.G. of the surface.

If we take a cubic foot as the unit of volume, the weight of a cubic foot of a fluid whose specific gravity is s, is s times the weight of a cubic foot of water, that is, is $62{\cdot}5 \times s$ pounds. Hence the pressure on a surface of area A immersed in such a fluid is

$$= 62{\cdot}5 \times s\mathrm{A}d \text{ pounds weight.}$$

If a cubic centimetre is taken as the unit of volume, the weight of a cubic centimetre of the fluid will be s grammes, so that the whole pressure on the surface of A square centimetres is

$$s\mathrm{A}d \text{ grammes weight.}$$

Example 1.—Find in grammes weight the amount of pressure on a rectangular area 8 centimetres long and 5 centimetres broad, its highest and lowest corners being at depths of 4 centimetres and 6 centimetres respectively in mercury of density 13·6.

The C.G. of the immersed area is the middle point of the line joining the highest and lowest corners, and is therefore at a depth of 5 centimetres below the surface of the mercury. Since the density of mercury is 13·6, the weight of a cubic centimetre of mercury is 13·6 grammes. The number of square centimetres in the immersed area is 8 × 5 or 40. Hence the total pressure on the area is

$$13{\cdot}6 \times 40 \times 5 = 2720 \text{ grammes weight.}$$

EXAMPLE 2.—Two cylindrical vessels of the same diameter and with vertical sides contain quantities of water and mercury respectively which produce the same pressure on the base of each. Taking the specific gravity of mercury at 13·6, compare the total pressures on the sides.

Let a denote the height in feet of the vessel containing mercury; then, since the pressures on the bases are the same, it is evident that the height of the other vessel must be $13{\cdot}6a$. Let r denote the radius in feet of either cylinder, then the pressure on the side of the water cylinder is

$$\frac{13{\cdot}6\,a}{2} \times 2\pi r \times 13{\cdot}6a \times 62{\cdot}5 \text{ pounds.}$$

Similarly the pressure on the side of the mercury cylinder is

$$\frac{a}{2} \times 2\pi r \times a \times 13{\cdot}6 \times 62{\cdot}5 \text{ pounds.}$$

Hence the pressure on the side of the water cylinder : the pressure on the side of mercury cylinder

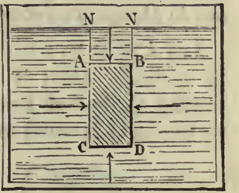
$$: : \frac{13{\cdot}6\,a}{2} \times 2\pi r \times 13{\cdot}6a \times 62{\cdot}5 : \frac{a}{2} \times 2\pi r \times a \times 13{\cdot}6 \times 62{\cdot}5$$
$$: : 13{\cdot}6 : 1;$$

that is, the pressure on the side of the water cylinder is 13·6 times the pressure on the side of the mercury cylinder.

148. Before proceeding to show how the specific gravities of substances are found experimentally we must first prove two important propositions.

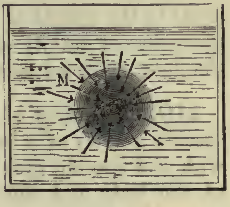
Prop. I. *When a body is completely immersed in a fluid it is pressed upwards with a force equal to the weight of the fluid which it displaces.*

Let ABCD be any body immersed to any depth in a fluid. Then the fluid pressures on this body will have as their resultant a single upward force equal to the weight of the fluid displaced. This is seen immediately by imagining the body to be replaced by the fluid which it displaces, and by supposing this fluid to become solid. The pressures of the surrounding fluid on this solidified portion will be the same as on the body ABCD. This solidified portion will be at rest under the action of two forces only: (1) its own weight downwards, (2) the pressure of the surrounding fluid. For equilibrium these must be equal and opposite, and it therefore follows that the resultant of the fluid pressures on the solidified portion is a single upward force equal to the weight of that portion. Hence it follows also that the resultant of the fluid pressures on the body ABCD is a single upward force equal to the weight of the fluid displaced.



In the above figure we have supposed the body to be of rectangular section, but the reasoning is perfectly general and will apply equally well to the case in which the body immersed is not bounded by plane surfaces.

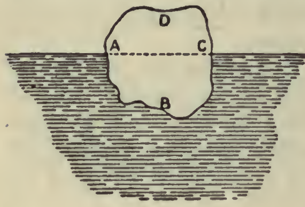
It follows that if the specific gravity of the body is just equal to that of the fluid, the body will float in equilibrium at any depth at which it may be placed. If the specific gravity of the body is greater than that of the fluid, the weight of the body would be greater than the weight of the fluid it displaced, and would therefore be greater than the upward pressure of the surrounding fluid. Consequently the body would sink in the fluid. Lastly if the specific gravity of the body is less than that of the fluid, the weight of the body will be less than the weight of the fluid it displaces, and consequently the body will rise in the fluid.



This Proposition is often referred to as the **principle of Archimedes.**

Prop. II. *When a body floats partly immersed in a fluid the weight of the body is equal to the weight of the fluid displaced, and the centres of gravity of the body and of the fluid displaced are in the same vertical line.*

Let ABCD be a body floating with the part ACB immersed in a fluid. Then it is easily seen by reasoning precisely similar to that of Prop. I., that the resultant fluid pressure on the body is equal to the weight of the fluid ACB displaced, and that this fluid pressure must act through the centre of gravity of ABC, the fluid displaced. The only other force acting on the body is the weight. Hence the body is in equilibrium under the action of the two forces, the weight of the body acting vertically downwards through the C.G. of the body, and the fluid pressure equal to the weight of the fluid displaced acting vertically upwards through the centre of gravity of the fluid displaced. For equilibrium the weight of the body must be equal to the weight of the fluid displaced and the line joining the centre of gravity of the body and the centre of gravity of the fluid displaced must be in the same vertical line.



149. By means of Proposition II., we can compare the specific gravity of a body floating partly immersed in a fluid with the specific gravity of the fluid if the whole volume of the body and the volume of the part immersed can be determined.

Let v_1 be the whole volume of the body and s_1 its specific gravity; let v_2 be the volume of the part immersed and s_2 the specific gravity of the fluid. Then if w denote the weight of unit volume of water, the weight of the body is wv_1s_1, and the weight of the fluid displaced is wv_2s_2. These are equal; hence

$$wv_1s_1 = wv_2s_2,$$

from which
$$\frac{s_1}{s_2} = \frac{v_2}{v_1}.$$

EXAMPLE 1.—Determine the weight and the specific gravity of a cubic foot of wood, which, when floating in water, is immersed to a depth of $9\frac{1}{2}$ inches.

Here $$s_2=1, \quad v_2=\frac{9\frac{1}{2}}{12}\times\text{cubic foot},$$

$$v_1=1 \text{ cubic foot;}$$

hence $$s_1=\frac{9\frac{1}{2}}{12}=\frac{19}{24}.$$

Also the weight of a cubic foot of wood whose specific gravity is $\frac{19}{24}$ is

$$=\frac{19}{24}\times 62{\cdot}5=49{\cdot}5 \text{ pounds nearly.}$$

EXAMPLE 2.—The specific gravity of lead is 11·4, and that of cork is ·24. How much lead must be attached to a piece of cork weighing 3 lbs. to make it just sink in water?

As many pounds of lead must be attached to the cork as will make the weight of the lead and the weight of the cork together equal to the weight of a volume of water equal to the volume of the lead and cork together.

Let x represent the number of pounds of lead required. Then a volume of water equal to the volume of x pounds of lead would weigh $\frac{x}{11{\cdot}4}$ pounds, and a volume of water equal to the volume of 3 pounds of cork would weigh $\frac{3}{{\cdot}24}$ pounds.

Hence the weight of a volume of water equal to the sum of the volumes of the lead and cork is

$$\left(\frac{x}{11{\cdot}4}+\frac{3}{{\cdot}24}\right) \text{ pounds.}$$

Now this must be equal to the weight of the lead and cork together, that is, this must be equal to

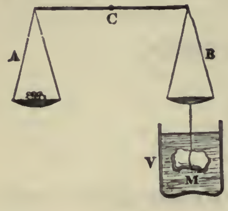
$$(x+3) \text{ pounds.}$$

Hence we obtain the equation

$$x+3=\frac{x}{11{\cdot}4}+\frac{3}{{\cdot}24}.$$

Solving this we find

$$x=10{\cdot}4 \text{ lbs. nearly.}$$



150. If a body M be suspended in a fluid by a fine thread from B, one of the scale-pans of a balance ACB, the weights which must be put in the scale-pan A to maintain a balance is said to be the weight of the body M in the fluid EF. From Prop. I. of Article 148 it follows that the weight of the body will be less than the weight of the body in air by the weight of fluid displaced by the body M.

151. Method of finding the Specific Gravity of a Solid which is not acted on Chemically by Water.

I. *Let the solid be specifically heavier than water.*

Let w denote the weight of the body in air,
w' „ „ water,

then $w-w'$ is the weight of a volume of water equal to the volume of the body. Hence by the definition of specific gravity, the specific gravity of the body is

$$\frac{w}{w-w'}.$$

II. *Let the body be specifically lighter than water.*

We attach to the body a piece of some dense material such as lead, called the sinker, to make the body sink in water.

Let w denote the weight of the body in air,
x „ „ sinker in air,
w' „ „ body and sinker in water,
y „ „ sinker in water,

then $w+x$ denotes the weight of body and sinker in air, and $w+x-w'$ denotes the weight of a volume of water equal to the volume of the body and sinker. Also $x-y$ denotes the weight of a volume of water equal to the volume of the sinker; hence $w+x-w'-(x-y)$ or $w-w'+y$ denotes the weight of a volume of water equal to the volume of the body. Hence the specific gravity is

$$\frac{w}{w-w'+y}.$$

We may arrive at this result in another way:—w' is the weight of the body and sinker in water and y is the weight of the sinker in water; therefore $w'-y$ is the weight of the body in water. Now $y>w'$, so that $w'-y$ is a negative quantity. It is, in fact, the force required to keep the body from rising in the water. Taking $w'-y$ as the weight of the body in water, we have $w-(w'-y)$ or $w-w'+y$ as the weight of water displaced by the body, and the formula for the specific gravity follows as before.

The student will notice that it is not necessary to know x, the weight of the sinker in air.

In weighing a body in air we ought to take into account that the body loses weight equal to the weight of the air displaced. As, however, the weight of a given volume of air is only about $\frac{1}{800}$ of the weight of the same volume of water, it is not necessary in ordinary determinations of specific gravity to make a correction for the weight of air displaced in weighing the body in air. The formulæ of this Article will be strictly true if we take w to represent the weight of the body in vacuo, that is, the weight of the body increased by the weight of the air which it displaces when it is weighed in air.

Example 1.—The specific gravities of gold and silver are 19·4 and 10·5 respectively. An alloy of these two metals weighs 979 ounces in vacuo and 890 in water. Find the quantities of gold and silver in the alloy.

The specific gravity of the alloy is

$$\frac{979}{979-890}=\frac{979}{89}=11.$$

Let w_1 and w_2 denote the quantities of gold and silver in the alloy; then by formula of Article 146,

$$\begin{aligned}\frac{w_1}{19{\cdot}4}+\frac{w_2}{10{\cdot}5}&=\frac{979}{11}\\&=89.\end{aligned}$$

Also $$w_1+w_2=979.$$

Solving these two simultaneous equations for the two unknowns w_1 and w_2, we get

w_1, the weight of gold = 97 ounces,
w_2, " silver = 882 ounces.

Example 2.—A body weighs 150 grains in air and when immersed in water by means of a sinker, the joint weight is 36 grains. The sinker alone weighs 75 grains in water. Find the specific gravity of the body.

Here $w=150$, $w'=36$, $y=75$.

Hence the specific gravity is

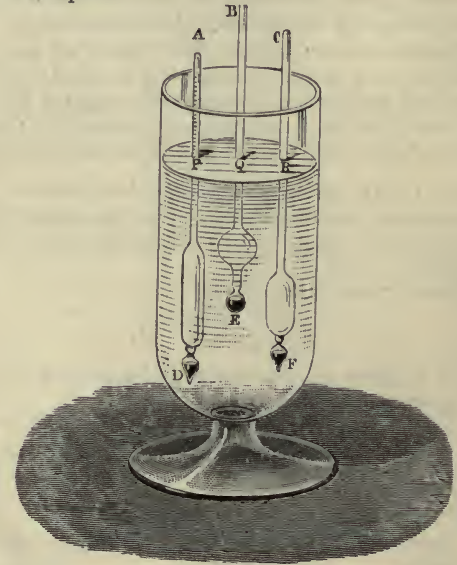
$$\frac{150}{150-(36-75)}=\frac{150}{189}=\frac{50}{63}.$$

152. Instruments used for the Determination of Specific Gravities.

Instruments of various forms are in use for the practical determination of the specific gravity of substances.

(1) The Common Hydrometer.

This instrument is used in the determination of the specific gravity of a liquid.

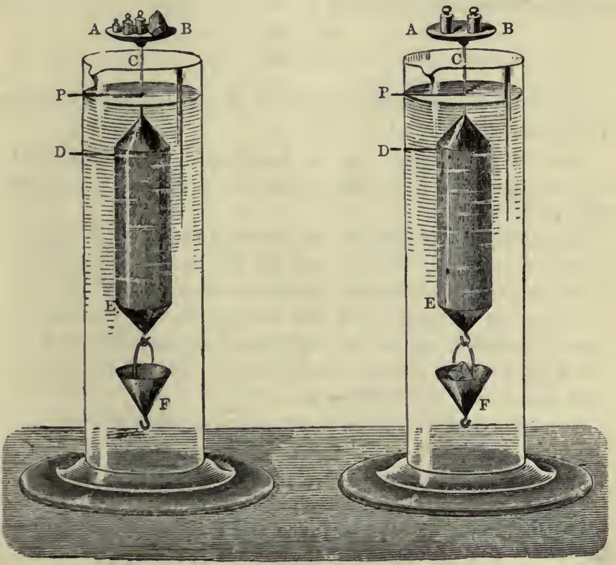


In the accompanying figure are shown three forms AD, BE, CF of hydrometers. A hollow cylinder or ball of glass is fitted above with a long slender stem, A, B or C, and below with a short stem terminating in a bulb, D, E or F, filled with mercury. When the instrument is placed in a liquid, the mercury serves to keep it floating vertically.

The long stem is graduated by the maker in such a way that when the instrument is placed in a liquid the reading on the stem at the point, P, Q or R, to which the instrument sinks, gives the specific gravity of the liquid.

(2) Nicholson's Hydrometer.

This instrument consists of a hollow brass cylinder DE from the upper part of which rises a stem C of steel, about $\frac{1}{40}$th of an inch in diameter. This stem supports a dish AB, while an iron stirrup fixed to the lower part of the cylinder supports another dish F. A fine mark is cut in the stem C at some point P.



The instrument may be used to find the specific gravity of a liquid or of a solid.

(*a*) *To Find the Specific Gravity of a Liquid by Nicholson's Hydrometer.*

Let W denote the weight of the instrument, W_1 and W_2 the weights to be placed in the dish AB to make the instrument sink to the point P respectively in water and in the liquid whose specific gravity is being found.

Then

weight of water displaced by instrument is $W + W_1$,
" liquid " $W + W_2$,

and since the volume displaced is the same in both cases, the specific gravity of the liquid is

$$\frac{W + W_2}{W + W_1}.$$

(*b*) *To Find the Specific Gravity of a Solid by Nicholson's Hydrometer.*

Let the instrument float in water, and let W denote the weight placed in AB which causes the instrument to sink to the point P.

Removing the weight W, let the solid be placed in dish AB as in the left-hand figure, and let W_1 be the weight which must be added to make the instrument sink to the point P.

Removing the solid and the weight W_1, let the solid be placed in dish F, as in the right-hand figure, and let W_2 be the weight which must be placed in the dish AB to make the instrument sink to the point P.

From the first two determinations we see that the weight of the solid in air is $W - W_1$. From the first and third determinations we see that the weight of the solid in water is $W - W_2$. (If the specific gravity of the solid is less than unity, this will be a negative quantity.) Hence the weight of a volume of water equal to the volume of the solid is $W - W_1 - (W - W_2)$ or $W_2 - W_1$, and therefore the specific gravity of the solid is

$$\frac{\text{weight of solid}}{\text{weight of equal vol. of water}} = \frac{W - W_1}{W_2 - W_1}.$$

(3) Specific Gravity Bulbs.

These are small glass bulbs of different specific gravity, the specific gravity of each being determined by experiment and marked on the surface. They are principally used for finding the specific gravity at different points of a liquid which is not of uniform density. If a number of these bulbs is thrown into such a liquid, each bead will come to rest at the point of the liquid where the specific gravity of the liquid is equal to that of the bead.

EXAMPLES XXX.

1. In a certain state of the atmosphere 100 cubic inches of air weigh 31 grains, and at the same temperature as that of the air 30 cubic inches of mercury weigh 14·88 pounds. Find the number of cubic inches of air which contain as much matter as a cubic inch of mercury.
2. Under certain conditions of pressure and temperature 100 cubic inches of oxygen weigh 35 grains, and at the same temperature a cubic inch of mercury weighs ·49 of a pound. How many cubic inches of the oxygen would contain as much matter as a cubic inch of mercury?
3. One body A has a volume of 1·35 cubic feet and a specific gravity of 4·4. A second body B has a volume of 10·8 cubic inches and a specific gravity of 19·8. What ratio does the quantity of matter in A bear to that in B?
4. Find the volume and specific gravity of a body which weighs 256 grammes in air and 192 in water.
5. A solid weighs 100 grammes in air, 64 in water and 60 in salt and water. Find the specific gravity of the salt and water.
6. A glass bottle is weighed empty, filled with water, and filled with nitric acid. The weights in the three cases are 4·32, 10·84 and 14·1 grammes respectively. Find the specific gravity of nitric acid.
7. A cylinder the base of which is 6 inches in radius, and which is 14 feet high, is filled with mercury whose specific gravity is 13·6. Find the pressure on the base and the pressure on the curved surface.
8. If two pints of water and 3 pints of alcohol are mixed, find the specific gravity of the mixture, assuming that there is no contraction. Specific gravity of alcohol is ·8.

9. A vessel, the base of which is a horizontal right-angled triangle such that the sides which contain the right angle are 6 centimetres and 9·1 centimetres long respectively, and that the sides are rectangles 15 centimetres long, is filled with alcohol. (Sp. gr. ·8.) Find the pressures on base and sides.

10. Find the total pressure on every square inch of the bottom of the sea where it is a mile deep, the specific gravity of sea water being 1·0275.

11. In what proportions (1) by weight, (2) by volume, must two liquids of specific gravities s and s' be mixed in order that the sp. gr. of the mixture may be the arithmetic mean between s and s'?

12. A mixture formed of equal weights of three fluids is divided into three parts, and to each part is added its own weight of one of the fluids. The sp. gr. of the mixtures so formed are as 3 : 4 : 5. Find the ratios of the sp. gr. of the three fluids.

13. A rectangular vessel 12 inches deep, 10 inches long, and 7 inches wide is half filled with water, and then filled to the top with oil of sp. gr. ·864. Find the pressure in pounds weight upon the bottom, and upon each of the vertical sides—the weight of a cubic inch of water being 250 grains.

14. Two vertical cylindrical vessels of the same size and communicating with each other at the bottom, are filled with water to a height of two feet. Half the water in one is then removed and replaced by oil of sp. gr. ·7. Show how high the water will stand in each vessel when equilibrium has been restored.

15. A body whose weight is 12 lbs. and sp. gr. 2·5 is placed in a vessel with a horizontal base containing water. What pressure is exerted on the base by the body, supposing the body to be quite covered with water?

16. A triangular lamina ABC floats vertically in a liquid with the angle B in the surface, and the angle A not immersed. Prove that AC is vertical, and compare the densities of liquid and lamina.

17. Is it advantageous for a trader to use weights of a dense material or not?

18. An ounce of silver, sp. gr. 10·5, is suspended in water. Find the tension of the string.

If the vessel and water originally weighed 30 ounces, what weight will counterbalance them after the ounce of silver is suspended in the water?

19. Find the cross section of a cylinder 120 centimetres long, which weighs 250 grammes in air and 200 grammes in water.

20. A rod of uniform cross section 27 inches long weighs 3 lbs., its sp. gr. being 3·6. What is the area of its cross section, a cubic inch of water being taken to weigh 250 grains?

21. A piece of copper wire 4 metres long weighs 1720 grammes in air and 1520 grammes in water. Find the diameter of the wire.

22. Find the volume and density of a body which weighs 128 grammes in air and 106 grammes in brine of density 1·1.

23. If a pound of iron and a pound of lead are both immersed in water, which of them will require the greater force to sustain it, and why?

What will these forces be if the sp. gr. of iron be 9, and that of lead be 11?

24. The force required to keep one of two bodies submerged in alcohol (sp. gr. ·8) is the same as that necessary to support another when immersed in it. The volumes of the bodies are 11 and 9 cubic centimetres respectively, and the weight of the first is 12·1 grammes. Find the sp. gr. of the bodies.

25. A cylindrical piece of basalt (sp. gr. 3), the height of which is 6 inches and radius 7 inches, is floating in mercury (sp. gr. 13·6). What force will be required to submerge it?

26. A uniform cylinder floats in water with one inch of its length above the surface. In a liquid whose sp. gr. is 1·5 it has 3 inches above the surface. What is the length of the cylinder? What is its specific gravity?

27. A cubic inch of lead of sp. gr. 11·4 is attached to a cubic foot of ice of sp. gr. ·92. Find whether the combination will float or sink in water.

28. A mass of 150 lbs. and sp. gr. 1·1, just floats in water by the help of a quantity of cork. Find in cubic feet the volume of cork necessary for this purpose, the sp. gr. of cork being ·24.

29. A cylinder of wood weighing 10 lbs. floats in water with $\frac{2}{3}$ of its bulk immersed. When 6 lbs. of metal are tied to it, the two together can just float. Find the sp. gr. of the wood and of the metal.

30. A cube floats in a liquid of uniform density with one angle below the surface and 3 in the surface. Show that the sp. gr. of the fluid is 6 times that of the cube.

31. Pieces of ice are floating in water, and gradually melt. How will the surface of the water be affected?

32. Find the specific gravity of a substance which weighs 5 grammes in air and 3·5 grammes in a liquid whose specific gravity is ·8.

33. A quantity of pebbles weighing $111\frac{1}{3}$ lbs. is put into a vessel of 1 cubic foot capacity, which then requires 540 cubic inches of water to fill it completely. Find the weight of a cubic foot of the stone of which the pebbles consist, and also find the sp. gr. of the stone.

34. A solid weighs 120 grammes in air and 90 grammes in a liquid of sp. gr. ·9. Find the sp. gr. of the solid.

35. The weight of a piece of wood is 4 lbs., and of a piece of lead in water 4 lbs., and of the lead and wood in water 3 lbs. What is the sp. gr. of the wood?

36. Find the sp. gr. of a piece of cork from the following data:—

Weight of cork in air = 2 grammes.
Weight of cork and sinker in water = 4 grammes.
Weight of sinker in water = 12 grammes.

37. Find what weight of lead attached to 20 lbs. of cork would be just sufficient to sink it in water. Sp. gr. of lead, 11·4; sp. gr. of cork, ·24.

38. A cork weighs in air 15 grains. The sinker weighs in water 90 grains, and the cork and sinker together weigh in water 45 grains. Required the sp. gravity of the cork.

39. A substance A weighs 10 grains in water, and a substance B weighs 14 grains in air. A and B connected together weigh 7 grains in water. What is the sp. gr. of B?

40. A cubic foot of cork weighing 250 ounces floats in water. What pressure (acting through the centre of gravity) is required to entirely immerse it, taking the sp. gr. of the cork to be $\frac{1}{4}$?

41. A man who weighs 12 stones can just float in water with a certain quantity of cork attached to him. Find the volume, assuming that the man's specific gravity is 1·12, that of cork ·24.

42. A body weighs a grammes in water and b grammes in a liquid of sp. gr. s. Find its true weight and volume.

43. A substance of which the specific gravity is 6 weighs 180 grains in air and 155 in a liquid. Required the sp. gr. of this liquid.

44. A solid weighs 100 grains in water and 115 in a liquid of which the specific gravity is ·8. What is the weight of the solid?

45. A brass ball weighing 12 oz., and having a sp. gr. of 8, hangs by a fine thread in oil of sp. gr. ·92. Find the tension of the thread.

46. A body of density ·8 weighing 100 grammes in air is attached to a lead sinker weighing 50 grammes in water, and the two are completely immersed in water. What weight is required in the opposite scale for equilibrium?

47. If the same body have weights W_0, W_1, W_2, ... in vacuo and in various liquids respectively; compare the sp. gr. of these liquids.

48. A hydrometer is found to sink to points marked A, B, C on its stem when placed in water and in two other liquids (β and γ say) respectively, the specific gravity of γ being ·9. If B is half-way between A and C, what is the sp. gr. of the liquid β?

49. A hydrometer floats in a liquid sp. gr. ·9 with 5 inches more of its stem above the surface than it would have if placed in water. What point of the stem will be on the surface when it floats in a liquid whose sp. gr. is ·95?

CHAPTER XIV.—GASES.

153. Gases.

Gases are fluids which tend to expand indefinitely, so that, unlike liquids, they cannot be kept in an open vessel. Gases differ from liquids also in their compressibility. Water may be taken as the type of liquids, and common air as the type of gases. Water exhibits great resistance to compression (Art. 130), while air can be compressed indefinitely by the application of sufficient pressure.

Since gases are forms of matter, they are acted on by forces, and therefore have weight. This is proved experimentally for the case of air by weighing a closed vessel first when containing air, and second when the air has been withdrawn from the vessel by means of an air-pump. It is found that the first weight is greater than the second.

The mass of a cubic centimetre of air at the surface of the earth is about ·001293 of a gramme, that is, about $\frac{1}{800}$th of a gramme.

154. Pressure of a Gas in a Closed Vessel.

Since a gas has weight, it follows from the principles of hydrostatics that the pressure of a gas on a unit of area of the surface will vary from point to point of the surface. As the weight of a gas, however, is small, it is usual to neglect the differences of pressure at different points of the surface due to the weight of the gas, and to consider the pressure on every unit of area of the surface to be the same.

If we know the pressure on any unit of area of the surface, we call that the **pressure of the gas.**

Thus we speak of the pressure of steam in a boiler as being so many (say 100) pounds per square inch. Strictly speaking we ought to say that the pressure of the steam in the boiler, at the point where the pipe connecting the boiler with the steam-gauge leaves the boiler, is 100 pounds per square inch.

155. Pressure of the Atmosphere.

In the case of the atmosphere we have an example of a gas not contained in a closed vessel. In this case the gas is prevented from expanding indefinitely by the force of gravity, which attracts it to the surface of the earth. It is probable that the atmosphere does not extend beyond a height of 50 miles.

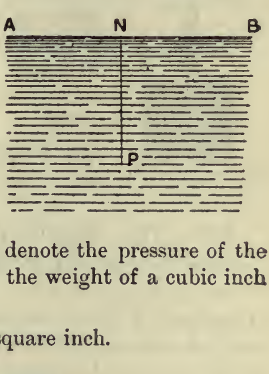
At the surface of the earth the atmosphere presses with a force of about 15 pounds on every square inch of every body. This pressure is due to the weight of the column of air of section one square inch, extending from the surface of the earth to the limits of the atmosphere.

In experiments in which very great pressures are used, as in experiments on the compressibility of liquids, the atmospheric pressure is taken as the unit, and is spoken of as the **pressure of an atmosphere.** As the pressure of the atmosphere at the surface of the earth varies from place to place and from time to time at the same place, the pressure of an atmosphere is not an invariable unit of pressure. It has been proposed recently to use as the pressure of an atmosphere the pressure of a million dynes per square centimetre. This pressure does not differ much from the pressure of 15 pounds per square inch, and has the advantage of being absolutely invariable.

156. Pressure at a Point of a Liquid.

In Article 136 we saw that the pressure at any point P in a liquid was equal to the weight of a column of liquid of height PN standing on unit area.

Now we have seen that the atmosphere exerts a pressure downwards on the surface of every body. The atmospheric pressure on the surface AB of the liquid will be communicated equally to all parts of the liquid. Thus, to take account of the pressure of the atmosphere, we must add on to the pressure found as above a correction for the pressure of the atmosphere.



Thus, if the square inch is taken as the unit of area, and if p denote the pressure of the atmosphere per square inch, and w the weight of a cubic inch of the fluid, the pressure at P

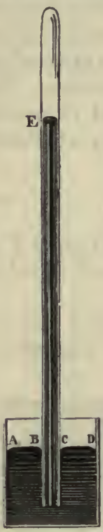
$$=(p+w.\mathrm{PN}) \text{ per square inch.}$$

157. The Mercurial Barometer.

A barometer is an instrument for measuring the pressure of the atmosphere.

In constructing a mercurial barometer a glass tube more than 32 inches long, open at the end, and closed at the other end, is held in a vertical position and filled with mercury. The tube is then inverted, the open end being closed with the finger to prevent the escape of the mercury, and is placed in a vertical position in a basin of mercury

ABCD, with the open end below the surface of the mercury. On removing the finger from the open end the mercury will subside from the top of the tube and the surface will descend to some point E from 28 to 31 inches above the surface ABCD of the mercury in the basin. The space above E will be a vacuum.



The column of mercury BE is supported by the pressure of the atmosphere on the surface AD of the mercury, which is communicated to the section BC of the barometer. This is clearly shown experimentally by placing the tube and the basin of mercury under the receiver of an air-pump (Art. 162), and exhausting the air from the receiver. As the air is withdrawn from the receiver, the pressure is diminished and the mercury gradually falls in the tube. On admitting the air again to the receiver the mercury rises to its former level E.

The height BE to which the mercury stands in the tube is called **the height of the barometer.***

158. The Height of the Barometer is Proportional to the Atmospheric Pressure.

In the figure of the preceding Article let the area of the cross section BC of the barometric tube be A square inches, and let p denote the atmospheric pressure in pounds on every square inch of the surface of the mercury in the basin.

The pressure of the atmosphere on A square inches of the surface of the mercury is pA pounds, and it is this pressure which balances the weight of the mercury in BE. The volume of this mercury is

$$\text{BE} \times \text{A cubic inches}$$

$$= \frac{\text{BE} \times \text{A}}{12 \times 12 \times 12} \text{ cubic feet.}$$

---

* 30 inches, or 76 centimetres, or 760 millimetres, is generally adopted as the standard height of the barometer.

Hence, if s denote the specific gravity of mercury, the weight of mercury in BE is

$$\frac{\text{BE} \times \text{A} \times 62{\cdot}5 \times s}{12 \times 12 \times 12} \text{ pounds.}$$

Hence $$p\text{A} = \frac{\text{BE} \times \text{A} \times 62{\cdot}5 \times s}{12 \times 12 \times 12};$$

from which $$p = \frac{\text{BE} \times 62{\cdot}5 \times s}{12 \times 12 \times 12}.$$

Hence p varies as BE, the height of the barometer.

Example 1.—What is the pressure of the atmosphere on a square inch if the height of the barometric column is 30 inches?

The specific gravity of mercury is 13·6, and a cubic foot of water weighs 62·5 lbs.

The pressure per square inch is equal to the weight of a column of mercury one square inch in section, of a height equal to 30 inches. Hence the pressure of the atmosphere

$$= \frac{30}{12 \times 12 \times 12} \times 62{\cdot}5 \times 13{\cdot}6 \text{ lbs.}$$
$$= 14{\cdot}8 \text{ lbs. nearly.}$$

Example 2.—If a barometer were constructed with water instead of mercury, to what height would the water rise in the tube if the height of the mercurial barometer were 30 inches?

The water would rise to such a height that the weight of the column of water would be equal to the weight of a column of mercury 30 inches in height.

Since the specific gravity of mercury is 13·6, the pressure of the atmosphere will support a column of water 13·6 times the height of the column of mercury. Hence the height of the water barometer would be

$$= 30 \times 13{\cdot}6 \text{ inches.}$$
$$= 34 \text{ feet.}$$

159. Variations in the Height of the Barometer.

The height of the barometer varies (1) from time to time at the same place, (2) from place to place at the same time.

If a barometer is carried up a mountain it is found that the height of the mercury in the barometer becomes less the higher the height ascended. Hence the higher the height the less is the pressure of the atmosphere. This is, of course, what we should expect, as the pressure per square inch at any station exceeds the pressure at any higher station by the weight of a column of air one square inch in section, extending from the lower to the upper station.

For small heights the fall in the barometer is, roughly speaking, proportional to the height ascended. Thus, for the first few hundred feet the fall is about $\frac{1}{10}$th of an inch for every hundred feet. But this rule cannot be applied with any degree of accuracy to a considerable height. It is found both by theory and experiment that at heights which form a series in Arithmetical Progression the heights of the barometer form a series in Geometrical Progression, the common ratio of this Geometrical Progression being less than unity. Thus if there are three stations A, B and C, at different levels, the height of B being midway between the heights of A and C, the fall of the barometer between B and C is less than the fall between A and B.

A formula has been investigated from theory by which the difference of heights of two stations can be found when we know (1) the heights of the barometers, (2) the temperatures of the atmosphere, and (3) the accelerations of gravity at the two stations.

160. Boyle's Law.

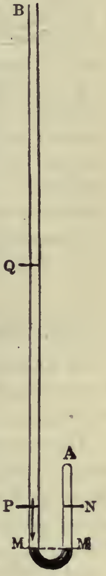
The pressure of a given quantity of a gas varies inversely as the volume of the gas when the temperature is constant.

The truth of this law for common air was discovered independently by two natural philosophers, Boyle in England and Marriott in France, who lived in the latter part of the 17th century. It was afterwards found to be true for all gases by Gay Lussac.

The law is proved experimentally for common air by the following experiment:—

A bent tube AB, with the parts AM and BM straight and parallel, is taken. Into this tube, which is open at both ends, there is poured a little mercury so as to fill the curved portion

MM, and the end A is then closed. We have then imprisoned in MA a quantity of air at the pressure of the atmosphere as indicated by the height of the barometer at the time and place of the experiment.



Mercury is again poured into the tube at the open end B of the tube until the level of the mercury in the closed branch AM of the tube rises to the point N midway between A and M. Suppose that Q is the point at which the mercury stands in the open branch when the mercury has risen to N in the closed branch. Let P be the point on the open branch on a level with N.

Since N and P are on the same level the pressure at N is equal to the pressure at P. Now the pressure at P is equal to the pressure of the column of mercury PQ together with the atmospheric pressure on the open surface Q. Hence the pressure at N, that is, the pressure of the air confined in AN, is equal to the sum of the atmospheric pressure and the pressure of the column PQ of mercury.

Now it is found that PQ is equal to the height of the barometer, that is, the pressure at P or at N is double the atmospheric pressure. Hence the air, which under atmospheric pressure occupied the space AM, under double the atmospheric pressure occupies the space AN, which is just half the space AM.

Hence by doubling the pressure the volume is halved. In the same way by increasing the pressure to three times the atmospheric pressure, we should find that the air in AM is compressed to one-third of its original bulk, and so on. Hence the volume varies inversely as the pressure.

Since the volume of a given mass of a gas varies inversely as the density, the law can also be expressed in the form:—

The density of a given quantity of a gas varies as the pressure of the gas when the temperature is constant.

Example.—A vessel of 3 cubic feet capacity containing air at two atmospheres' pressure is put into communication with a vessel of 18 cubic feet capacity containing air at $\frac{1}{4}$ atmospheric pressure. What is now the pressure of air in the two vessels?

The air in the first vessel, if allowed to expand until the pressure is equal to the atmospheric pressure, would occupy 2×3 or 6 cubic feet. The air in the second vessel, if compressed until the pressure was equal to the atmospheric pressure, would occupy $\frac{1}{4} \times 18$ or $4\frac{1}{2}$ cubic feet. Hence the air in the two vessels would at atmospheric pressure occupy $(6 + 4\frac{1}{2})$ or $10\frac{1}{2}$ cubic feet. When the two vessels are put into communication this air actually occupies $(3 + 18)$ or 21 cubic feet. The quantity of air, therefore, which would occupy $10\frac{1}{2}$ cubic feet at atmospheric pressure, occupies 21 cubic feet. Hence the pressure of the air is

$$\frac{10\frac{1}{2}}{21} = \frac{1}{2} \times \text{atmospheric pressure.}$$

161. Height of Homogeneous Atmosphere.

It follows from the second form of Boyle's law that, since the pressure of the atmosphere decreases with the height, the density must also decrease with the height.

For some purposes in physics a uniform atmosphere is conceived which would be of the density of the air at a given point, and of such a height as to exert a pressure equal to the actual atmospheric pressure at the given point. Such an atmosphere is called the **homogeneous atmosphere** at the point. By the use of the height of the homogeneous atmosphere many formulæ in physics can be simplified.

At a given place the height of the homogeneous atmosphere depends on the temperature but not on the atmospheric pressure.

Let H denote the height of the homogeneous atmosphere, p the pressure of the atmosphere in pounds per square inch, and w the weight of a cubic inch of air at the density of the air at the place. Then a column of air of height H exerts a pressure p pounds on every square inch. Hence

$$p = w\text{H},$$

from which

$$\text{H} = \frac{p}{w}.$$

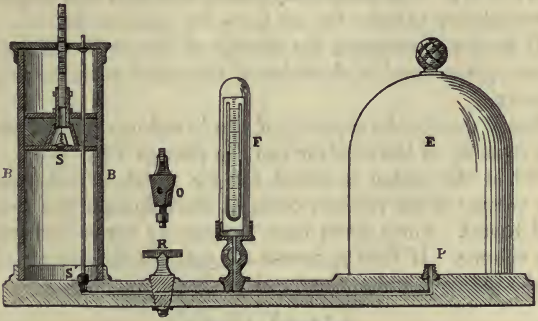
Now w is the weight of a cubic inch of air at the given place, and therefore varies as the density of air at the place. Hence H is proportional to the pressure divided by the density. But by the second form of Boyle's law the pressure is proportional to the density, so that pressure divided by density is constant.

Hence the height of the homogeneous atmosphere at a given place does not depend on the atmospheric pressure at the place. It increases with the temperature.

162. The Common Air-pump.

The air-pump is used for withdrawing the air from a vessel.

The vessel E, called the receiver, from which the air is to be exhausted, is placed mouth downwards on a plate in the centre of which is a small aperture P. From this aperture a passage, opening by a valve at S′, leads into a cylinder BB. In this cylinder there works a piston fitted at S with a valve opening upwards. The arrangement for working the piston upwards and downwards is not shown in the figure.



Suppose that the piston is being raised from the bottom of the cylinder. The valve S in the piston will be closed by the outside atmospheric pressure, and a vacuum will be left between the piston and the bottom of the cylinder. There will, therefore, be no pressure downwards on the valve S′, which will be pressed upwards by the air in the passage PS′. Consequently this valve will be raised and air will flow from the receiver E along the passage PS′ into the space SS′.

When the piston begins to descend the volume of air SS′ will be diminished, and consequently by Boyle's law the pressure will increase, and will soon exceed the pressure of air in the receiver E. The valve S′ will then close, and the pressure of air in SS′ will increase until it exceeds the atmospheric pressure, when the valve S in the piston will open upwards. As the piston is pressed down the air in SS′ will escape through the open valve S into the outer atmosphere. Thus in every upward stroke of the piston the cylinder or barrel will be filled with air from the receiver, and in the succeeding downward stroke this air will escape. As the machine is worked the air in the receiver will become more and more rarefied, and the action will go on until the pressure of air in the receiver is so slight that it is not sufficient to open the valve at S′. It is principally due to this that it is impossible to completely exhaust the air from the receiver E.

It is easy to compare the density of the air in the receiver after a given number of strokes of the piston with the original density.

Let V denote the capacity of the barrel or cylinder, and V′ the capacity of the receiver and the passage PS′.

When the piston is raised, the air which at first occupied the volume of the receiver occupies the volume of the receiver and barrel. For a given mass the density varies inversely as the volume. If then ρ_0 denote the original density of the air in the receiver, and ρ_1 the density after one stroke,

$$\rho_1 : \rho_0 = V' : V + V',$$

from which

$$\rho_1 = \rho_0 \frac{V'}{V + V'}.$$

Similarly, if ρ_2 denote the density after the second upward stroke

$$\rho_2 = \rho_1 \frac{V'}{V + V'} = \rho_0 \left(\frac{V'}{V + V'}\right)^2.$$

Generally, if ρ_n denote the density after the n^{th} upward stroke

$$\rho_n = \rho_0 \left(\frac{V'}{V + V'} \right)^n.$$

EXAMPLE.—The receiver of an air-pump has five times the volume which the barrel has, the stroke of the piston being 12 inches. How far must the piston descend after the third stroke before the valve in the piston will rise?

In this case $V' = 5$, $V = 1$, so that the density of the air in the barrel after the third stroke is

$$\rho_0 \left(\frac{5}{6} \right)^3 = \rho_0 \frac{125}{216},$$

where ρ_0 is the original density of air in the receiver. Now by Boyle's law the density varies as the pressure, and therefore the pressure of air in the barrel after the third stroke is $\frac{125}{216}$ of the atmospheric pressure. The valve will rise when the piston has descended so far that the pressure of this air becomes equal to the atmospheric pressure. (We neglect the weight of the valve.)

Let x be the required distance in inches through which the piston must descend; then

volume of air after descent of piston through the distance x
: volume of air before descent of piston
$= 12 - x : 12.$

But by Boyle's law the pressure is inversely proportional to the volume. Hence

$\frac{125}{216} \times$ atmospheric pressure : atmospheric pressure
$= 12 - x : 12,$

from which we get the equation

$$\frac{125}{216} : 1 = 12 - x : 12,$$

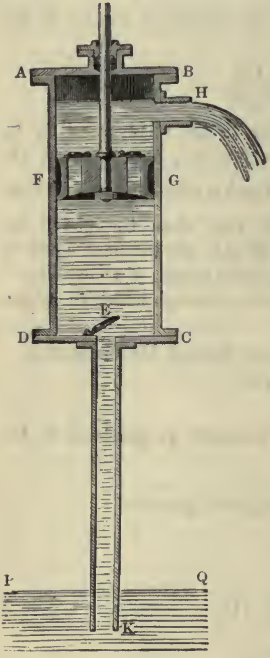
or

$$\frac{125}{216} = \frac{12 - x}{12}.$$

Solving for x, we find $x = 5\frac{1}{18}$ inches.

163. The Common or Suction Pump.

In the common pump a piston FG, in which there is a valve which opens upwards, works air-tight in a cylinder ABCD, at the lower part of which is an aperture fitted with a valve E, which opens upwards. From this aperture a pipe EK leads down to water in a well, the end K of the pipe dipping below the surface PQ of the water.



When the piston FG is drawn upwards the pressure of the atmosphere on the valve of the piston will close that valve and a vacuum will be left in the space DCGF. The pressure of the air in the pipe EK will open the valve E, and the air which at first occupied the volume of the pipe EK, will now occupy the volume of the pipe EK and of the space DCGF. The pressure of the air in the pipe EK will thus be reduced below the atmospheric pressure, and consequently the atmospheric pressure on the surface PQ of the well will force some water up the pipe EK. When the piston descends, the valve E will close, while the valve in the piston will open and allow the escape of the air in the space DCGF.

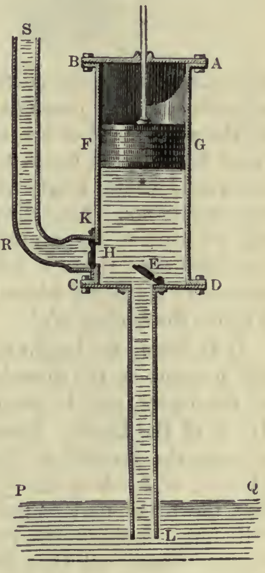
On continuing to work the pump the pressure of air in the pipe EK will become less and less, and the water in that pipe will rise higher and higher until it rises to E. So far the pump acts virtually as an air-pump. On still continuing to work the pump, the water will flow through the valve E on the upward stroke of the piston, will be forced upwards into the space FGAB through the valve in the piston on the downward stroke, and will issue from an aperture H in the side of the cylinder.

Since it is the pressure of the atmosphere which forces the water up the pipe KE into the cylinder, it is evident that the depth of the surface PQ of the well below DC must not be greater than the height of the water barometer. (See Example 2, Article 158.)

164. The Forcing Pump.

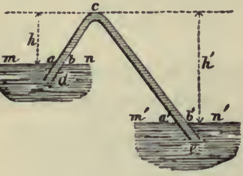
In this pump a solid piston FG works in a closed cylinder ABCD, between the limits B and K. The aperture H through which the water issues is below the point K, and is fitted with a valve which opens outwards. In other respects this pump does not differ from the common pump.

The explanation of the working of the pump is similar to that given of the working of the common pump. After the pump has been worked for some time the pressure of the atmosphere forces the water up the pipe LE to the valve E. If the piston is now raised the water will rise into the cylinder DCGF, and on the downward stroke of the piston the valve E closes, the valve H opens, and the water in DCGF is forced up the pipe RS. The valve H serves the purpose of preventing the return of the water in the pipe RS to the cylinder of the pump when the piston is on the upward stroke.



165. The Siphon.

The siphon consists of a bent tube *dce*, one branch *ce* of which is much longer than the other *dc*. It is used for the purpose of withdrawing lipuid from a vessel. The tube is first filled with liquid, and the ends *d* and *e* are closed. The siphon is then placed so that the end *d* of the shorter branch is below the surface of the liquid *m n*, while the longer branch *ce* is outside the vessel. The ends *d* and *e* are then opened, when a stream of the liquid flows through the siphon and issues at *e*.



The action of the siphon depends on the pressure of the atmosphere. Let us take a section of the tube at the highest point *c*, and let us consider the forces acting at *c* in the directions *ce* and *cd* when the ends *d* and *e* are opened. The pressure in the direction *ce* is the pressure of the atmosphere on the surface *mn* of the liquid, communicated through the liquid to *c*, *less* the weight of a column of the liquid whose height is equal to the height, h, of *c* above the surface *mn*. Also the pressure at *c* in the direction *cd* is the pressure of the atmosphere on the liquid at the end *e*, communicated through the liquid to *c*, *less* the weight of a column of the liquid whose height is equal to the height, h', of *c* above the surface *m′n′*.

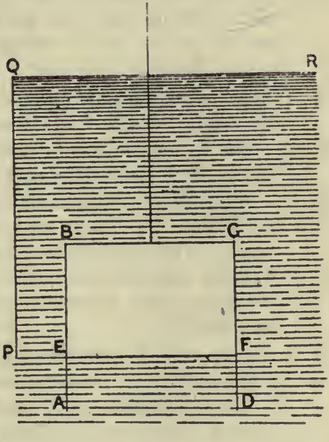
If H denote the height of a barometer filled with the liquid, the pressure in the direction *ce* is that due to a height $\mathrm{H}-h$ of the liquid, and in the direction *cd* that due to a height $\mathrm{H}-h'$ of the liquid. Hence the pressure in the direction *ce* exceeds the pressure in the direction *cd* by a height $\mathrm{H}-h-(\mathrm{H}-h')$ or $h'-h$ of the liquid. Hence the liquid at *c* will flow down the part *ce*, and the atmospheric pressure on *mn* will force liquid up the part *dc* to supply its place. A continuous stream will thus flow from *d* to *e*.

The siphon will continue to work as long as h' is greater than h, that is, as long as the level *m′n′* is lower than the level *mn*.

Also it is evident that the siphon will not work at all if h is greater than H, that is, if the height of *c* above *mn* is greater than the height of the barometer filled with the liquid.

166. The Diving-bell.

The diving-bell is a large vessel ABCD constructed of iron, closed at the top BC and at the sides AB and CD, but open at the bottom AD.



When the bell is let down into water, the air in the bell will tend to exclude the water. At any depth below the surface the air in the bell is subject to the pressure of the water, and is therefore compressed. As the bell descends, the pressure of the water increases, and therefore by Boyle's law the volume of air in the bell becomes less.

Let the bell be at such a depth that the level EF of the water in the bell is at a depth PQ below the surface. Then the air in the bell is under the pressure of the depth PQ of water *plus* the atmospheric pressure on the surface of the water. If h denote the height of the water barometer, the air in the bell may be considered to be under the pressure of a depth $(PQ + h)$ of water. Now before the bell was sunk in the water the air was under the pressure of a depth h of water. Hence by Boyle's law

$$\frac{\text{volume of air in bell at depth PQ}}{\text{volume of the bell}} = \frac{h}{PQ + h}.$$

EXAMPLES XXXI.

1. Taking the standard atmospheric pressure as a pressure of a million dynes per square centimetre, find the work done in ergs against atmospheric pressure when a piston accurately fitting the sides of a cylinder is drawn up so as to leave a vacuum measuring 240 cubic centimetres.

2. Taking the specific gravity of mercury at 13·6, find what would be the height of an oil barometer, specific gravity of the oil being ·845, when the mercury barometer stands at 29 inches.
3. A vertical tube closed at the top is full of water, and its lower end stands in water the surface of which is $8\frac{1}{2}$ feet below the top of the tube. What is the pressure of the water against the top of the tube?
4. Taking the atmospheric pressure at 15 lbs. per square inch, and the weight of a cubic foot of water at 62·5 lbs., find the pressure on the bottom of a tank filled with water, 2 feet deep, the area of the bottom of the tank being 2 square yards.
5. The height of a barometric tube above the level of the mercury in the basin is a. Some air has got into the vacuum space at the top of the tube, so that the mercury stands at h when a standard barometer reads H. What would be the reading of the standard barometer when the mercury in the tube stands at h'?
6. A quantity of air confined in a graduated tube over mercury measures 20 cubic centimetres when the surface of mercury in the tube stands at 46 centimetres above the surface of mercury in the basin. When the surface of mercury in the tube stands at 10·8 centimetres higher than the mercury in the basin, the volume of the confined air is 9 cubic centimetres.

   Show what is the height in centimetres of the mercurial barometer at the time.
7. A U-shaped tube of uniform bore is sealed at one end. Mercury is poured in so as to rise to the same height on each side of the bend, and to confine in the closed limb a quantity of air occupying 10 inches. The tube is then put under the receiver of an air-pump, and the pump is worked until the air in the tube occupies a length of 12 inches. Show what is now the pressure of the air in the receiver and in the tube respectively; the original pressure having been equal to that of 30 inches of mercury.
8. The mercury stands at the same level in the open and in the closed branch of a bent tube of uniform section—both branches being vertical—when the air confined at the closed end is at a pressure of 30 inches of mercury, which is that of the external air. Express in atmospheres the pressure which, acting on the surface of the mercury in the open branch, compresses the con-

fined air to one-third of its original volume and at the same time maintains a difference of 6 inches in the levels of the two mercurial columns.

9. A tube 17 inches long, closed at the top, dips vertically into a basin of mercury. The mercury outside rises to the middle point of the tube, and inside it stands 6 inches higher, leaving the rest of the tube filled with air. How far must the tube be drawn out, that the contained air may expand so as to fill another half inch?

10. A bubble of air, one inch in diameter, rises from the bottom of the sea 260 metres deep. Find the size of the bubble on reaching the surface if the barometer reads 760 millimetres. The specific gravities of sea-water and of mercury may be taken respectively to be 1·02 and 13·6.

11. An open vessel contains one gramme of air at a place where the barometer stands at 750 millimetres. It is then taken to a place where the barometer stands at 700 millimetres. What is now the mass of air in the vessel?

12. A cylinder is fitted with a piston which works in it easily and air-tight. When the water barometer stands at 33 feet, the piston is 1 foot above the bottom of the cylinder. If the cylinder were slowly sunk in water to a depth of 44 feet, how high would the piston be above the bottom of the cylinder?

13. A barometer stands at 30 inches, the vacuum above the mercury being perfect. The area of the cross section of the tube is $\frac{1}{4}$ square inch. If a quarter of a cubic inch of air is now allowed to get into the barometer and if the column of mercury falls 4 inches, what was the volume of the original vacuum?

14. A tube of uniform bore, stopped at one end and of a length equal to the height of the barometer, is put vertically with its open end downwards into mercury. At what depth must the open end be below the surface of the mercury if the mercury fills $\frac{1}{4}$th part of the tube?

15. The pressure of air in the receiver of an air-pump is a pressure of 30 inches of mercury. After a few strokes it is 27 inches. What part of the original contents of the receiver has been withdrawn?

16. The mercury in a barometer stands at 30 inches, and has a vacuum of 7 inches above the mercury. The cross section of the tube may for convenience be taken as 1 square inch in area. If a cubic inch of the external air is let into the tube so as to fill the vacuum space, how many inches will the mercury in the tube fall?

17. A vessel containing water is being emptied by means of a siphon. What effect would be produced by making a small hole through the top of the siphon?

18. If the piston or plunger in the forcing pump has a cross section of 4 square inches and works 80 feet below a cistern, with what pressure must it be pushed down in order to force water into the cistern?

19. In the common pump one foot of the length of the barrel of the pump holds a gallon of water, which may be taken to weigh 10 lbs. At each stroke the piston works through 8 inches. The spout is 24 feet above the surface of the water in the well. How many foot-pounds of work are done in each stroke?

20. A diving bell $7\frac{1}{2}$ feet high is let down into water until its top is $7\frac{1}{2}$ feet below the surface of the water. To what height will the water rise in the bell, the water barometer standing at 32 feet?

21. If the pressure in the receiver of an air-pump were reduced to $\frac{1}{3}$ of the atmospheric pressure in 4 strokes, to what pressure would it be reduced in 6 strokes?

22. A barometer is placed under the receiver of an air-pump, the mercury standing at 30 inches before the pump is worked. If one stroke of the pump brings it down to 20 inches, where will it stand after 3 strokes?

23. The mercury in a barometer taken down in a diving-bell is found to stand 18 inches higher below water than it did above the surface. Show how far the diving-bell must have sunk.

Take the specific gravity of mercury at 13·6.

24. The volume of the barrel of an air-pump is $\frac{1}{8}$ of that of the receiver, and the pressure of the air in the latter is a pressure of 765·45 millimetres of mercury. To what will this be reduced after 3 strokes of the pump?

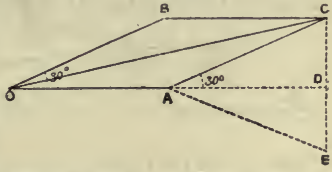
APPENDIX.

167. Particular Cases of the Parallelogram Construction.

In Article 20 we gave a general formula, obtained with the help of trigonometry, for the resultant of two component velocities inclined at any angle. We shall now show how the value of the resultant may be found without the use of trigonometry in the six particular cases in which the angle between the components is (1) 30°, (2) 45°, (3) 60°, (4) 120°, (5) 135°, (6) 150°. The case in which the angle is 90° (Article 19) does not require trigonometry, so that there are in all seven particular cases in which the resultant can be found by elementary geometry and algebra. Of course the formulæ which we shall obtain will apply equally to velocities, accelerations, and forces.

In each case we take OA and OB as the two components; then, on completing the parallelogram OACB, the resultant is represented by OC. It is required to find OC in terms of OA and OB.

(1) When the angle AOB is 30°. Draw CD the perpendicular on OA produced, and produce CD to E, making DE = CD. Then the triangle ADE is equal in all respects to the triangle DAC. Therefore the angle EAD is equal to the angle DAC, which is equal to the angle AOB, or 30°. Hence the whole angle EAC is 60°, and is therefore equal to the angle ACD, the complement of the angle CAD. Hence the triangle ACE is equilateral, and CD = ½ CE = ½ CA. Hence, by Euclid I. 47,



$$AD^2 = AC^2 - CD^2 = AC^2 - \tfrac{1}{4}AC^2 = \tfrac{3}{4}AC^2;$$

from which $$AD = \frac{\sqrt{3}}{2}AC.$$

Now by Euclid II. 12,

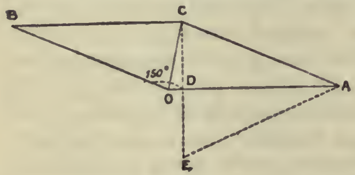
$$OC^2 = OA^2 + AC^2 + 2\,OA\,.\,AD,$$

or
$$OC^2 = OA^2 + AC^2 + 2\,OA\,.\,\frac{\sqrt{3}}{2}\,AC,$$

or
$$OC^2 = OA^2 + OB^2 + \sqrt{3}\;OA\,.\,OB, \text{ (since } AC = OB)$$

the required formula.

(2) When the angle AOB is 150°, the supplement of 30°.



We make the same construction as in the preceding case, and as before

$$AD = \frac{\sqrt{3}}{2}\,AC.$$

Hence, by Euclid II. 13,

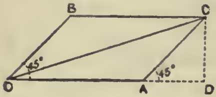
$$OC^2 = OA^2 + AC^2 - 2\,OA\,.\,AD,$$

or
$$OC^2 = OA^2 + AC^2 - 2\,OA\,.\,\frac{\sqrt{3}}{2}\,AC,$$

or
$$OC^2 = OA^2 + OB^2 - \sqrt{3}\,.\,OA\,.\,OB, \text{ (since } AC = OB)$$

the formula required.

(3) When the angle AOB is 45°. Draw CD perpendicular to OA produced. Then the triangle CAD is isosceles, each of the angles CAD and ACD being 45°. Hence AD = DC, and therefore by Euclid I. 47,



$$2\,AD^2 = AC^2,$$

from which $AD = \frac{1}{\sqrt{2}}\,AC.$

Hence by Euclid II. 12,

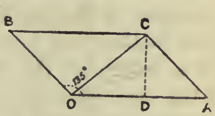
$$OC^2 = OA^2 + AC^2 + 2OA\,.\,AD,$$

or
$$OC^2 = OA^2 + AC^2 + 2OA\,.\,\frac{1}{\sqrt{2}}\,AC,$$

or
$$OC^2 = OA^2 + OB^2 + \sqrt{2}\;OA\,.\,OB, \text{ (since } AC = OB)$$

the formula required.

(4) When the angle AOB is 135°, the supplement of 45°. Drawing CD perpendicular to OA we have, as in the preceding case,



$$AD = \frac{1}{\sqrt{2}} AC.$$

Hence by Euclid II. 13,

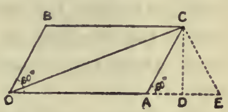
$$OC^2 = OA^2 + AC^2 - 2\,OA\,.\,AD,$$

or $$OC^2 = OA^2 + AC^2 - 2\,OA\,.\,\frac{1}{\sqrt{2}}\,AC,$$

or $$OC^2 = OA^2 + OB^2 - \sqrt{2}\,OA\,.\,OB, \text{ (since } AC = OB)$$

the required formula.

(5) When the angle AOB is 60°. Draw CD perpendicular to AE, take DE in AD produced equal to AD, and join CE. Then it easily follows that the triangle CAE is equilateral, and therefore



$$AD = \tfrac{1}{2}\,AE = \tfrac{1}{2}\,AC.$$

Hence by Euclid II. 12,

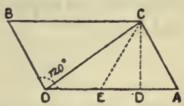
$$OC^2 = OA^2 + AC^2 + 2\,OA\,.\,AD,$$

or $$OC^2 = OA^2 + AC^2 + 2\,OA\,.\,\tfrac{1}{2}\,AC,$$

or $$OC^2 = OA^2 + OB^2 + OA\,.\,OB, \text{ (since } AC = OB)$$

the required formula.

(6) When the angle AOB is 120°, the supplement of 60°. Drawing CD perpendicular, and taking DE in DO equal to DA, it follows, as in the preceding case, that the triangle CEA is equilateral, and that



$$AD = \tfrac{1}{2}\,CA.$$

Hence, by Euclid II. 13,

$$OC^2 = OA^2 + AC^2 - 2\,OA\,.\,AD,$$

or $$OC^2 = OA^2 + AC^2 - 2\,OA\,.\,\tfrac{1}{2}\,CA,$$

or $$OC^2 = OA^2 + OB^2 - OA\,.\,OB, \text{ (since } AC = OB)$$

the required formula.

168. Solutions by Graphical Constructions.

The student will find it an excellent exercise to work examples of the parallelogram construction by actual drawing. This method has been referred to on page 30, where the use of a protractor for measuring angles is recommended. In many cases the angles can be obtained by simple geometrical constructions, and in these cases the only instruments that are required for accurate drawing are a ruler, a pair of compasses, and a scale of inches or millimetres. Thus to find the resultant of two components 3 and 5 inclined at an angle of 60°, we construct an angle of an equilateral triangle AOB, and along the two lines containing this angle we measure lengths OA and OB respectively 3 and 5 units (inches or millimetres). We complete the parallelogram OA CB, and join OC. If the diagram is carefully drawn OC will be found, on being measured, to contain almost exactly 7 units (inches or millimetres), so that the resultant of the two components is 7. This is the value which the formula (5) of the preceding Article would give.

The student will not be able to obtain perfectly accurate results by drawing, but, if the diagram is carefully drawn, the error in any case will not exceed 1 per cent of the true result. Errors will arise chiefly from inaccurate constructions of angles. A right angle, 90°, can be constructed almost exactly; as also may 60°, the angle of an equilateral triangle, and therefore its supplement 120°. By bisecting a right angle we get 45°, and therefore its supplement 135; and by bisecting 45° we get 22°½, and so on. By bisecting 60° we get 30°, and therefore its supplement 150°, and by bisecting 30° we get 15°, and so on.

EXAMPLES XXXII.

1. Find the resultant in each of the following cases in which two components are given:—

(1) 15 and 16 at 90°.
(2) 17 and 18 at 60°.
(3) 21 and 23 at 45°.
(4) 25 and 27 at 30°.
(5) 17 and 18 at 150°.
(6) 11 and 13 at 60°.
(7) 47 and 51 at 135°.

2. Find by accurately drawn diagrams the resultant, to one decimal place, in each of the following cases:—

(1) 7 and 8 at 90°.
(2) 9 and 10 at 45°.
(3) 15 and 16 at 30°.
(4) 19 and 21 at 60°.
(5) 99 and 20 at 90°.
(6) 3 and 5 at 60°.
(7) 15 and 23 at 120°.
(8) 17 and 19 at 150°.
(9) 25 and 27 at 135°.
(10) 10 and 13 at 15°.
(11) 12 and 17 at 7°½.
(12) 15 and 29 at 22°½.
(13) 25 and 31 at 165°.

3. Given in each of the following cases the resultant, one component, and the angle between the resultant and that component, find by accurately drawn diagrams the other component:—

(1) Resultant, 18; component, 17; angle, 30°.
(2) „ 19; „ 23; „ 60°.
(3) „ 23; „ 25; „ 90°.
(4) „ 29; „ 47; „ 150°.
(5) „ 41; „ 51; „ 120°.
(6) „ 43; „ 55; „ 45°.
(7) „ 11; „ 11; „ 7°½.
(8) „ 17; „ 19; „ 15°.

4. The ends of a string are tied to two points A and B in the same horizontal line, and at a point C of the string a mass of 100 lbs. is hung on. Given in each of the following cases the lengths of AB, BC, and CA, find (Art. 49) by accurately drawn diagrams the tensions of the two parts of the string AC and BC:—

(1) AB = BC = CA = 12 feet.
(2) AB = 12 feet; BC = CA = 13 feet.
(3) AB = 15 feet; BC = 8 feet; CA = 9 feet.
(4) AB = 13 feet; BC = 14 feet; CA = 15 feet.
(5) AB = 8 feet; BC = 7 feet; CA = 9 feet.

5. A mass B of 50 lbs. is hung on at the end B of a string AB, which is attached to a fixed point A. A second string is attached to the mass, and is pulled with a horizontal tension F. Find by construction (Art. 49) the value of F and the tension in the first string AB when AB is inclined to the vertical at an angle of — (1) 7°½, (2) 15°, (3) 22°½, (4) 30°, (5) 45°, (6) 60°, (7) 67°½, (8) 75°, (9) 82°½.

169. Modulus of Elasticity.

Elasticity is resistance to change of shape or volume. When the shape or volume of a body is changed, the amount of the change is called the **strain**, and the force producing the strain is called the **stress.**

A solid body may be strained in several ways: it may be stretched, compressed, twisted, or bent. It is found by experiment that, within certain limits, *the strain produced is proportional to the stress.* Thus if the stress is doubled, the strain is doubled, and if the stress is halved, the strain is halved, and so on.

A **modulus** or **coefficient of elasticity*** of a body for a particular kind of strain is a number measuring the resistance the body exhibits to that kind of strain, and for every kind of strain is the value of the fraction

$$\frac{\text{stress}}{\text{strain}}.$$

Young's Modulus of Elasticity is a number measuring the resistance of a wire or rod to being strained by stretching. The stress may be supposed to be applied in the form of a weight hung on at one end of the wire, the other end of the wire being attached to a fixed point. In this case the strain is not the whole extension of the wire, but the extension per unit of length, that is, the whole extension divided by the original length of the wire. Dividing the stress by the strain, so calculated, the quotient will give Young's modulus of elasticity for that wire.

EXAMPLE.—A weight of 30 lbs. is hung on at the end of a wire 25 feet long, and the wire is thereby stretched $1\frac{1}{2}$ inches. What is Young's modulus of elasticity for that wire?

$$\text{The stress} = 30 \text{ lbs.},$$

$$\text{and the strain} = \frac{1\frac{1}{2}}{12} \div 25 = \tfrac{1}{200}.$$

Hence the value of Young's modulus is

$$30 \div \tfrac{1}{200} = 6000.$$

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* A modulus of elasticity must not be confounded with a coefficient of restitution (Art. 116), which was formerly, and, in some text-books, is still called a coefficient of elasticity.

170. Energy of Rotation.

If a body is rotating round an axis, the energy of rotation is

$$\tfrac{1}{2}\omega^2 Mk^2,$$

where ω represents the angular velocity of the body round the axis and Mk^2 the moment of inertia of the body about the given axis.

This may be proved thus:—

Imagine the body broken up into particles whose masses are $m_1, m_2, m_3, \ldots$ and whose distances from the axis are $r_1, r_2, r_3, \ldots$ respectively. The velocities of these particles are (Art. 91) $r_1\omega, r_2\omega, r_3\omega, \ldots$ respectively, and their kinetic energies are (Art. 107) $\tfrac{1}{2}m_1(r_1\omega)^2, \tfrac{1}{2}m_2(r_2\omega)^2, \tfrac{1}{2}m_3(r_3\omega)^2, \ldots$ respectively. Hence the total kinetic energy of the body is

$$\tfrac{1}{2}m_1(r_1\omega)^2 + \tfrac{1}{2}m_2(r_2\omega)^2 + \tfrac{1}{2}m_3(r_3\omega)^2 + \ldots$$

or $$\tfrac{1}{2}m_1r_1^2\omega^2 + \tfrac{1}{2}m_2r_2^2\omega^2 + \tfrac{1}{2}m_3r_3^2\omega^2 + \ldots$$

or $$\tfrac{1}{2}\omega^2(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots)$$

The part of this expression in brackets represents the sum of all the products obtained by multiplying the mass of each particle by the square of its distance from the axis, and this sum by Art. 98 is the moment of inertia of the body about the axis. Since Mk^2 represents the moment of inertia, it follows that the total energy of the rotating body is

$$\tfrac{1}{2}\omega^2 \times Mk^2.$$

If M is expressed in pounds and k in feet, this expression will give the energy in foot-poundals. If M is expressed in grammes and k in centimetres, this expression will give the energy in ergs.

Example.—A fly-wheel, 8 feet in diameter, is rotating 3 times in a second. If we suppose the whole mass of the fly-wheel, 15 cwt., to be distributed round the circumference, find in foot-pounds the energy the wheel has in virtue of its rotation.

Here
$$M = 15 \times 112,$$
$$k^2 = (\text{radius})^2 = 4^2,$$
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\frac{1}{3}} = 6\pi;$$

hence the energy of rotation in foot-poundals is,

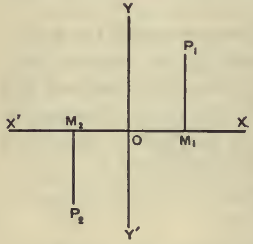
$$\tfrac{1}{2}\, 15 \times 112 \times 4^2 \times (6\pi)^2,$$

and in foot-pounds is (dividing by g)

$$\frac{15 \times 112 \times 4^2 \times 36\pi^2}{2 \times 32}$$
$$= 151200 \qquad \text{(taking } \pi^2 = 10).$$

171. Co-ordinates.

Let $P_1, P_2, P_3, \ldots$ be the positions of any number of points in the same plane, and let X′OX and Y′OY be straight lines in the plane drawn at right angles through a chosen point O. From the points $P_1, P_2, P_3, \ldots$ let straight lines, $P_1M_1, P_2M_2, P_3M_3, \ldots$ be drawn perpendicular to X′OX. Then the position of any one of the points (P_1 for example) will be known with reference to the lines X′OX and Y′OY when we know—



(1) the lengths of the lines OM_1 and M_1P_1; (2) the direction in which OM_1 is measured from O, that is, whether it is measured along OX towards the right from O, or along OX′ towards the left from O; (3) the direction in which M_1P_1 is measured from M_1, that is, whether it is measured from M_1 upwards in the direction of OY, or downwards in the direction of OY′.

The distances OM_1 and M_1P_1 are together called the **co-ordinates** of the point P_1; OM_1 is called the **abscissa**, and M_1P_1 the **ordinate** of P_1. The position of P_1 will thus be completely determined when we know its co-ordinates and the directions in which those co-ordinates are measured. In accordance with the conventions introduced in Trigonometry (see the Author's Trigonometry, chap. iv.), a length such as OM_1, measured along OX towards the right from O, is reckoned positive, and is represented by a positive number; and a length such as OM_2, measured along OX′ towards the left from O, is reckoned negative, and is represented by a negative number. Similarly a line, as M_1P_1, drawn upwards from X′OX, is reckoned positive, and is represented by a positive number; and a line, as M_2P_2, drawn downwards from X′OX, is reckoned negative, and is represented by a negative number.

From these conventions it follows that—

(1) A point in the first quadrant, that is, within the angle XOY, has its abscissa +, its ordinate +.

(2) A point in the second quadrant, that is, within the angle YOX′, has its abscissa −, its ordinate +.

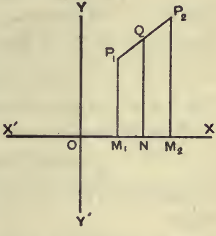
(3) A point in the third quadrant, that is, within the angle X′OY′, has its abscissa −, its ordinate −. P_2 is such a point.

(4) A point in the fourth quadrant, that is, within the angle Y′OX, has its abscissa +, its ordinate −.

The point whose abscissa is a and ordinate b is often referred to as the point whose co-ordinates are (a, b), or simply as the point (a, b).

The lines X′OX and Y′OY are called the **axes of co-ordinates**: X′OX is called the axis of x, Y′OY the axis of y.

EXAMPLE.—Mark in a diagram the positions of the four points whose co-ordinates are (2, 1); (−1, 1); (−1, −2); (2, −2); and show that these four points are the four corners of a square.



172. Co-ordinates of the Centre of Inertia of a given System of Masses.

Let P_1, P_2, P_3, . . . be the positions of masses m_1, m_2, m_3, . . . and let (x_1, y_1), (x_2, y_2), (x_3, y_3), . . . be the co-ordinates of $P_1, P_2, P_3, \ldots$ Let $(\bar{x}, \bar{y})$ be the co-ordinates of the centre of inertia of the masses.

The centre of inertia of the masses m_1 at P_1 and m_2 at P_2 is (see Art. 67) a point Q in the line P_1P_2 dividing the line P_1P_2 inversely as the masses. Let x denote ON, the abscissa of Q; then noticing that by geometry $P_1Q/P_2Q = M_1N/NM_2$, we have

$$m_1/m_2 = P_2Q/P_1Q = NM_2/M_1N = (OM_2 - ON)/(ON - OM_1)$$
$$= (x_2 - x)/(x - x_1),$$

from which $\quad m_1(x - x_1) = m_2(x - x_2)$,

giving $\quad x(m_1 + m_2) = m_1x_1 + m_2x_2$.

The centre of inertia of the three masses m_1, m_2, m_3 is a point (R suppose) in QP_3 dividing that line inversely as $m_1 + m_2 : m_3$. Let x' denote the abscissa of R; then we find x' from x and x_3 by the formula which has just been proved for x. Thus

$$x'\{(m_1 + m_2) + m_3\} = (m_1 + m_2)x + m_3x_3.$$

But $\quad (m_1 + m_2)x = m_1x_1 + m_2x_2$;

therefore $\quad x'(m_1 + m_2 + m_3) = m_1x_1 + m_2x_2 + m_3x_3$.

Similarly the abscissa (x'' say) of the centre of inertia of four masses m_1, m_2, m_3, m_4 will be given by

$$x''(m_1 + m_2 + m_3 + m_4) = m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4,$$

and the abscissa of the centre of inertia of *any* number of masses will be found from the formula—

$$\bar{x}(m_1 + m_2 + m_3 + \ldots\ldots\ldots) = m_1x_1 + m_2x_2 + m_3x_3 + \ldots\ldots\ldots$$

For shortness write

Σmx for $m_1x_1 + m_2x_2 + m_3x_3 + \ldots\ldots\ldots\ldots\ldots$

and $\quad \Sigma m$ for $m_1 + m_2 + m_3 + \ldots\ldots\ldots\ldots\ldots$;

then $\quad \bar{x}\Sigma m = \Sigma mx$,

or $\quad \bar{x} = \Sigma mx/\Sigma m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

Similar working to the above will give $\bar{y}$. We shall leave the student to verify that

$$\bar{y} = \Sigma my/\Sigma m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Formulæ (1) and (2) give the co-ordinates of the centre of inertia.[*]

[*] If the masses are not all in the same plane we take a third axis OZ at right angles to the plane XOY, and the position of the C. I. is determined by three co-ordinates $\bar{x}, \bar{y}, \bar{z}$, where $\bar{x}$ and $\bar{y}$ are given by (1) and (2), and $\bar{z} = \Sigma mz/\Sigma m$.

Corollary. If we suppose that $m_1, m_2, m_3, \ldots$ are parallel forces instead of masses, then (see Articles 57 and 67) formulæ (1) and (2) are the co-ordinates of the *centre of parallel forces*. These formulæ are evidently independent of the direction of the parallel forces. We have thus a proof of the proposition stated on page 74.

Example 1.—ABCD is a square lamina whose weight is neglected. Masses of 4 lbs., 3 lbs., 2 lbs., and 1 lb. are placed at the angular points A, B, C, D respectively. Find the centre of inertia of the masses.

Take AB as axis of x and AD as axis of y, and let a represent the side of the square. Then the co-ordinates of the four masses are—

(1.) Of 4 lbs. at A, $(0, 0)$. (2.) Of 3 lbs. at B, $(a, 0)$. (3.) Of 2 lbs. at C, (a, a). (4.) Of 1 lb. at D, $(0, a)$.

Hence if $(\bar{x}, \bar{y})$ be co-ordinates of centre of inertia,

$$\begin{aligned}\bar{x} &= \Sigma mx/\Sigma m\\ &= (4\times 0+3\times a+2\times a+1\times 0)/(4+3+2+1)\\ &= 5a/10 = a/2,\end{aligned}$$

and

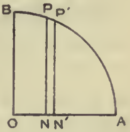
$$\begin{aligned}\bar{y} &= \Sigma my/\Sigma m\\ &= (4\times 0+3\times 0+2\times a+1\times a)/(4+3+2+1),\\ &= 3a/10.\end{aligned}$$

Hence the following construction for centre of inertia:—

Take E and F the mid points of AB and CD respectively; take EG three-tenths of EF; then G is the centre of inertia of the four masses.

Example 2.—Find the centre of inertia of a solid hemisphere.

Let AOB be a quadrant of a circle; then if the figure revolve round the line OA, it will generate a solid hemisphere of which OA will be the axis. Let a denote the radius OA.



Divide OA into n equal parts, each equal to d, so that $d = a/n$; let NN′ in the figure be one of these parts. When n is taken infinitely large, d will be an infinitesimal length. (Compare Art. 70.) In that case sections of the hemisphere through N and N′ will inclose what is practically an infinitely thin circular disc whose radius is PN and whose thickness is NN′ or d. If m denote the mass of this disc and M the mass of the whole hemisphere, we have—

$$\begin{aligned}m/\mathrm{M} &= \text{volume of disc}/\text{volume of hemisphere}\\ &= \pi.\mathrm{PN}^2.\mathrm{NN}'/\tfrac{2}{3}\pi a^3. \quad = 3d\mathrm{PN}^2/2a^3.\end{aligned}$$

Therefore $m = 3d.\mathrm{M}.\mathrm{PN}^2/2a^3.$

The centre of inertia of this disc will be at N, and therefore the disc may be considered as a particle of mass m placed at the point N. Similarly with all the other discs into which the hemisphere will be divided by planes drawn perpendicular to O A through the points of division on O A. Hence the centre of inertia of the hemisphere will lie on O A. Let $\bar{x}$ denote the distance of this point from O, and let x denote O N. Then

$$mx = 3d\,\mathrm{M}\,x\,\mathrm{PN}^2/2a^3 = 3d\,\mathrm{M}\,x\,(a^2 - x^2)/2a^3$$

substituting the value of m, and putting for PN^2 its equivalent $a^2 - x^2$.

By taking the sum of all such products obtained from the discs in succession, we get $\Sigma\, m\, x$.

Thus $$\Sigma\, m\, x = \Sigma\, 3d\,\mathrm{M}\,x\,(a^2 - x^2)/2a^3.$$

The expression on the right-hand side of this equation means that in the fraction $3d\,\mathrm{M}\,x\,(a^2 - x^2)/2a^3$ we put $x = d$, $x = 2d$, $x = 3d$, . . . $x = nd$, in succession, and then add all the results together. Taking out the factors common to each of these fractions we see, that

$$\Sigma\, m\, x = \frac{3d\,\mathrm{M}}{2a^3}\,\Sigma\, x\,(a^2 - x^2).$$

Now
$$\begin{aligned}\Sigma\, x(a^2 - x^2) &= \Sigma\,(a^2 x - x^3) = a^2\,\Sigma\, x - \Sigma\, x^3\\ &= a^2(d + 2d + 3d + \;.\;.\;.\; + nd)\\ &\quad - \{d^3 + (2d)^3 + (3d)^3 + \;.\;.\;.\; + (nd)^3\}\\ &= a^2 d.\frac{n(n+1)}{2} - d^3.\frac{n^2(n+1)^2}{4} \text{ (by Algebra).}\end{aligned}$$

Thus
$$\begin{aligned}\Sigma\, m\, x &= \frac{3d\,\mathrm{M}}{2a^3}\left\{ a^2 d.\frac{n(n+1)}{2} - d^3.\frac{n^2(n+1)^2}{4}\right\}\\ &= \frac{3\mathrm{M}}{2a^3}\left\{ \tfrac{1}{2}a^2 n^2 d^2\left(1 + \frac{1}{n}\right) - \tfrac{1}{4}n^4 d^4\left(1 + \frac{1}{n}\right)^2\right\}.\end{aligned}$$

Now $nd = a$, and since n is infinitely great, $1/n$ is zero. Thus

$$\begin{aligned}\Sigma\, m\, x &= \frac{3\mathrm{M}}{2a^3}\left\{ \tfrac{1}{2}a^4 - \tfrac{1}{4}a^4 \right\}\\ &= 3\mathrm{M}a/8.\end{aligned}$$

Also
$$\begin{aligned}\Sigma\, m &= \mathrm{M};\text{ and therefore}\\ \bar{x} &= \Sigma\, m x/\Sigma\, m = 3\mathrm{M}a/8\mathrm{M}\\ &= 3a/8.\end{aligned}$$

Hence the centre of gravity of a hemisphere is in the axis at a distance from the centre equal to three-eighths of the radius.

EXAMPLE 3. If (x_1y_1), (x_2y_2), (x_3y_3) be co-ordinates of the angular points of a triangle, the co-ordinates of the centre of inertia of the triangle will be

$$(x_1+x_2+x_3)/3 \text{ and } (y_1+y_2+y_3)/3.$$

EXAMPLE 4. P, Q, R, are parallel forces acting in the same direction at the angular points A, B, C, of an equilateral triangle; if P $=2$Q$=3$R, find the position of the centre.

Also find the position of the centre if the direction of the force Q is reversed. (Science and Art Examinations, 2nd stage, 1884.)

If $2a$ be length of side of triangle, and D the middle point of AB, the co-ordinates of the centre referred to DA and DC as axes of x and y are, in the *first* case $(3a/11,\ 2a\sqrt{3}/11)$, in the *second* case $(9a/5,\ 2a\sqrt{3}/5)$.

EXAMPLE 5. If masses 3 lbs., 7 lbs., 10 lbs., are respectively 5 feet above, 6 feet above, and 12 feet below a certain horizontal line, what is the position of the centre of gravity of these masses with reference to the horizontal line? (Science and Art Examinations, 2nd stage, 1884.)

Result—63/20 feet below horizontal line.

EXAMPLE 6. A rod AB of given length consists of two parts of equal cross sections joined end to end. The specific gravity of the part towards A is to that of the part towards B as 1 : 5. If the centre of gravity of the whole is 2/3 of the rod from A, what part of the length is of the heavier material? (Science and Art Examinations, 2nd stage, 1883.)

If x denote the fraction the heavier part is of the whole length, we get the quadratic equation

$$12x^2-8x+1=0,$$

giving $x=1/2$ or $1/6$. Hence two solutions.

EXAMPLE 7. Show by the method of Example 2 that the centre of inertia of a right circular cone is in the axis of the cone at a distance from the vertex equal to three-fourths of the length of the axis.

EXAMPLE 8. Assuming that the area of a zone of a sphere varies as its height, show that the centre of inertia of a hollow hemisphere is in the axis at a distance from the centre equal to half the radius.

EXAMPLE 9. Taking the case of motion discussed in Art. 80—the motion of a body projected in any direction not vertical—and selecting AB and AH (see figure, page 123) as axes of x and y, show that

(1) Co-ordinates of C are $(u^2\sin\alpha\cos\alpha/g,\ u^2\sin^2\alpha/2g)$,

(2) Co-ordinates of B are $(2u^2 \sin\alpha\cos\alpha/g,\ 0)$,

(3) The co-ordinates of the body t units of time after the moment of projection are $(u\,t\cos\alpha,\ u\,t\sin\alpha - \frac{1}{2}gt^2)$,

(4) The velocities, horizontal and vertical, t units of time after the moment of projection are respectively $u\cos\alpha$ and $u\sin\alpha - gt$,

(5) The direction of motion t units of time after the moment of projection makes with the positive direction of the axis of x an angle whose tangent is $(u\sin\alpha - gt)/u\cos\alpha$.

173. Solution of Statical Problems involving Friction by Geometrical Construction.—If R denote the normal pressure between two bodies which are just on the point of sliding, one over the other, and if F denote the friction, then

$$F = \mu R \quad\text{.............................}(1).$$

The forces R and F may be compounded into a single force (R′ say), acting in a direction inclined at an angle θ to the direction of R.

By Article 50, R′ and θ are determined by the equations

$$\begin{aligned} R' &= \sqrt{R^2 + F^2} \\ &= R\sqrt{1+\mu^2} \text{ (since } F = \mu R)\text{....}(2), \end{aligned}$$

and

$$\tan\theta = F/R = \mu \quad\text{.........................}(3).$$

Now if ϕ be the angle of friction, $\tan\phi$ is equal to μ. Hence equation (3) gives

$$\tan\theta = \tan\phi$$

or

$$\theta = \phi \quad\text{.............................}(4).$$

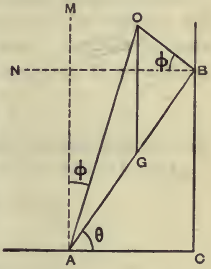
Calling R′ the *total* reaction between the two bodies, we can express result (4) thus—

When two bodies are just on the point of sliding, one over the other, the total reaction makes with the common normal to the surfaces an angle equal to the angle of friction.

The application of this principle will enable us to determine by simple geometrical construction the limiting positions of equilibrium of bodies under the action of three forces.

Example 1. AB, a uniform rod, is placed with one end A on a rough horizontal plane and the other end B on an equally rough vertical wall, the vertical plane through AB being perpendicular to the wall. It is required to find the limiting position of equilibrium.

Let the figure represent the position of the rod when just about to slide down, and let θ be the angle BAC which the rod makes in this position with the horizon.



If ϕ be the angle of friction, the direction AO of the total reaction at A makes an angle ϕ with the vertical AM, the angle MAO being measured from AM towards the right; and similarly the total reaction at B acts in a direction BO inclined to the horizontal line BN at an angle OBN equal to ϕ. The rod is just in equilibrium under the action of the total reactions at A and B and the weight of the rod acting vertically through G, the middle point of AB. Of these three forces two pass through O, and therefore the third must also pass through O. Hence the line OG must be vertical.

The solution is completed by geometry. In the triangle AOG we have by trigonometry

$$\begin{aligned} AG/GO &= \sin AOG/\sin OAG \\ &= \sin\phi/\sin(90^\circ - \theta - \phi) = \sin\phi/\cos(\theta + \phi), \end{aligned}$$

and in the triangle BOG

$$\begin{aligned} BG/GO &= \sin BOG/\sin OBG \\ &= \sin(90^\circ - \phi)/\sin(\theta + \phi) = \cos\phi/\sin(\theta + \phi). \end{aligned}$$

Now $\quad AG = BG, \quad$ and therefore $AG/GO = BG/GO$.

Hence $\quad \sin\phi/\cos(\theta + \phi) = \cos\phi/\sin(\theta + \phi);$

from which $\quad \tan\theta = (1 - \tan^2\phi)/2\tan\phi.$

By putting $\tan\phi = 1/3$, we get the problem in Ex. 1, Art. 85. On substituting $\tan\phi = 1/3$ in the equation just found, we get

$$\tan\theta = (1 - \tfrac{1}{9})/\tfrac{2}{3} = 4/3,$$

as in Art. 85.

Example 2. A body of given weight W is urged against a rough inclined plane by a force P acting in a direction at right angles to the plane. Under what circumstances will P just keep the body from sliding down the plane?

The inclination of the plane to the horizon being 30°, and the weight of the body 20 lbs., what must be the coefficient of friction between the body and the plane, if the force acting as above, which just keeps the body from sliding, is one of 10 lbs.? (Science and Art Examinations, 2nd stage, 1886.)

The resultant of P and W must act in a direction making with the direction of P an angle whose tangent is equal to μ, the coefficient of friction. Now the angle between P and W is equal to the inclination of the plane, which call i. Hence by Art. 50

$$\frac{\mathrm{W}\sin i}{\mathrm{P}+\mathrm{W}\cos i}=\mu,$$

the required relation between P, W, i, and μ.

If $\mathrm{P}=10$, $\mathrm{W}=20$, $i=30°$, we get

$$\begin{aligned}\mu&=\tfrac{1}{2}\times 20\Big/\left(10+20\times\frac{\sqrt{3}}{2}\right)\\&=10/(10+10\sqrt{3})=1/(\sqrt{3}+1)\\&=(\sqrt{3}-1)/2.\end{aligned}$$

Example 3. A is a smooth point at a given height a above a horizontal plane; a rod of uniform density, whose length is $2a$, is placed with one end on the plane and resting against the point. If it will stay at rest at any inclination greater than 60° to the horizon, find the coefficient of friction between the rod and the plane, and the pressures on the plane and on the point when the rod is just on the point of sliding. (Science and Art Examinations, 2nd stage, 1883.)

Coefficient of friction is $3(8+\sqrt{3})/61$; normal pressure on plane is $\mathrm{W}(2-\sqrt{3})/2$, and on point is $\mathrm{W}\sqrt{3}/4$, W being weight of rod.

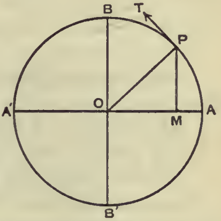
Example 4. Spheres, whose weights are W, W′, rest on different and differently inclined planes. The highest points of the spheres are connected by a horizontal string perpendicular to the common horizontal edge of the two planes and above it. If μ and μ', the coefficients of friction, are such that each sphere is on the point of slipping down, then $\mu\mathrm{W}=\mu'\mathrm{W}'$.

174. Simple Harmonic Motion.

Definition.—If a point P is describing the circumference of a circle with uniform velocity, and if PM be drawn perpendicular to A A′, a fixed diameter of the circle, then the motion of the point M in the line A A′ is called a **simple harmonic motion.**

It is evident that the motion of M will be an oscillatory motion. When P is at A, M will also be at A; and as P moves over the first quadrant A B, M will move along the radius A O, and will arrive at O when P arrives at B. While

P is describing the second quadrant BA′, M will move from O to A′, and P and M will again coincide at A′. Thus while P is describing the first half ABA′ of the circumference of the circle, M is moving from A to A′. Similarly, while P is describing the second half A′B′A of the circumference, M is moving from A′ back to A. Thus the motion of M is an oscillatory motion between the extreme points A and A′. We shall call the circle which P describes the *auxiliary circle* of the point M.



OA, or its equal OA′, is called the **amplitude** of the simple harmonic motion; so that the auxiliary circle is the circle described with centre O, the middle point of the harmonic motion, and with radius equal to the amplitude.

The **period** or **periodic time** of the simple harmonic motion is the time which elapses between successive passages of M through A. It is evidently equal to the periodic time in the auxiliary circle.

Let a be the amplitude and T the period; and let V and ω be the linear and angular velocities respectively in the auxiliary circle. We shall find expressions in terms of these letters for the velocity and acceleration of M in any given position. Let OM, the distance of M from O at any time, be denoted by x.

The **velocity** of M is the component of the velocity of P in the direction MO. Hence (Art. 23)

$$\begin{aligned}\text{velocity of M} &\\ &= \text{velocity of P} \times \text{cosine of angle PT makes with OA}\\ &= \text{V.sin AOP} = \text{V.PM}/a\\ &= \frac{\text{V}}{a}\sqrt{\text{OP}^2 - \text{OM}^2} = \frac{\text{V}}{a}\sqrt{a^2 - x^2} \quad \text{.......................(1).}\\ &= \omega\sqrt{a^2 - x^2} \quad \text{.......................(2).}\end{aligned}$$

since $\text{V}/a = \omega$ (Art. 91).

The **acceleration** of M is the component along the line OA of the acceleration of P. The acceleration of P is V^2/a along the radius PO (Art. 90), and therefore

$$\begin{aligned}\text{acceleration of M} &= (V^2/a).\cos AOP\\ &= (V^2/a).OM/a\\ &= (V^2/a^2).OM = (V^2/a^2)x \ldots\ldots\ldots(3).\\ &= \omega^2 x \ldots\ldots\ldots\ldots\ldots(4).\\ &= (4\pi^2/T^2)x \ldots\ldots(5).\end{aligned}$$

From the expressions for the velocity we deduce the results that

(*a*) when $x=a$, that is, at A or A′, the velocity is zero;

(*b*) when $x=0$, that is, at O, the velocity is V;

(*c*) the velocity is greatest at O.

The acceleration of M is always directed towards O. From the expressions for its value in any position we deduce the results that

(*d*) when $x=a$, that is, at A or A′, the acceleration is $V^2/a = \omega^2 a$;

(*e*) when $x=0$, that is, at O, the acceleration is zero;

(*f*) the acceleration is greatest at A or A′.

The general character of the motion is thus apparent. Starting from A with zero velocity and acceleration V^2/a along AO, the point M has a gradually increasing velocity and a gradually decreasing acceleration, both in the direction AO. At the moment of passing through O the velocity is V, and the acceleration is zero. In the second quarter of the period the body has an acceleration in the direction A′O, that is, in the direction opposite to that of the motion. With this negative acceleration the velocity of M falls from V at O to zero at A′. The motion in the third and fourth quarters of the period is simply the motion in the second and first quarters reversed in direction.

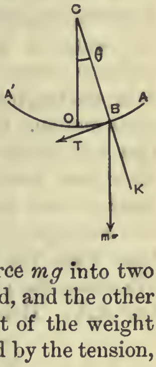
Calling OM the **displacement**, we can write the result (5) thus

$$\frac{\text{acceleration}}{\text{displacement}} = \frac{4\pi^2}{T^2}.$$

Corollary.—If a point M is oscillating in a straight line AA′ under an acceleration always directed towards O, the middle point of AA′, and equal to μ times the displacement from the middle position, then the motion of M is a simple harmonic motion, and the period is $2\pi/\sqrt{\mu}$.

For if on A A as diameter we describe a circle, and consider a point P to move in this circle with uniform angular velocity $\sqrt{\mu}$, then the projection N of P upon AA′ will describe the straight line AA′ with acceleration μ times distance ON. Thus if P and M start together from rest at A, it is evident that M and N will always coincide, since they start with the same velocity and have at each instant the same acceleration. But the motion of N is by definition a simple harmonic motion; therefore the motion of M is also a simple harmonic motion performed in a period equal to the period of N, that is, equal to the period of P, that is, in time $2\pi/\sqrt{\mu}$ (Art. 91).

175. Proof of the Formula for the Time of Vibration of a Simple Pendulum.



Let the figure represent a position of a simple pendulum of length l, in which the displacement of the bob M is the *arc* OB, which denote by x. The displacement in *angle* is the angle OCB, which denote by θ. The amplitude in *arc* is OA, which denote by a, and in *angle* is the angle OCA, which denote by α.

Consider the forces acting on the bob of the pendulum in this position. These are (1) the tension of the string in the direction CO, and (2) the weight mg vertically downwards, m being the mass of the bob. Resolve this force mg into two components, one along BK, the radius produced, and the other along BT, the tangent at B. The component of the weight along BK and the centrifugal force are balanced by the tension, while the component along BT is unbalanced. This component along BT is the only force acting in the direction BT, and it is therefore the force which gives the bob of the pendulum its acceleration. The value of this acceleration is by Newton's second law of motion this force divided by the mass of the bob of the pendulum. Thus the acceleration of bob of pendulum in the direction BT

$$= mg(\text{cosine angle TB makes with vertical})/m$$
$$= mg\sin\theta/m = g\sin\theta.$$

Now in the simple pendulum the angle θ is always a small angle, and we know from trigonometry that the sine of a small angle is very approximately equal to the radian measure of the angle (see author's Trigonometry, Art. 96). Hence the acceleration of the bob of the pendulum is very approximately

$$=g\theta=g.\text{OB}/\text{BC}=gx/l.$$

We have thus a case of motion in which the acceleration is proportional to the length x measured from the middle point of the motion.

Construct a circle having its radius equal to the arc OA, the amplitude of the motion of the pendulum; and let a point P describe this circle with uniform angular velocity $\sqrt{g/l}$. We may take the figure of Art. 175 to represent this circle. Then the projection M of P on AA′ will move in the line AA′ with acceleration equal to (g/l).OM, always directed towards O, the centre of AA′. Thus the acceleration of M when at a distance x from O will be equal to the acceleration of the bob B of the pendulum when at distance x from the middle point of its motion. It follows that if M start from A in the straight line AA′ at the instant when the bob of the pendulum is at its highest point A, the accelerations of M and B will be the same at that instant, and the velocities and accelerations at any subsequent instant of time will also be equal. It follows that while the bob of the pendulum is making a complete vibration, the point M is making a complete vibration in the line AA′. Hence the periodic time of the pendulum

= periodic time of M in AA′
= „ „ P in circle ABA′B′
= 2π/angular velocity of P (Art. 91)
$=2\pi/\sqrt{g/l}$
$=2\pi\sqrt{l/g}$.

Example 1. A particle is executing simple harmonic vibrations with an amplitude of 1/100 centimetre. If the time of one complete vibration is 1/200 second, find (1) velocity at middle point, (2) the velocity at point midway between the centre and an extreme point,

(3) the acceleration at an extreme point, (4) the acceleration at point midway between centre and extreme point.

Ans.—(1) 4π cm. per sec., (2) $2\pi\sqrt{3}$ cm. per sec., (3) $1600\pi^2$ cm. per sec. per sec., (4) $800\pi^2$ cm. per sec. per sec.

EXAMPLE 2. Show that the velocity of the bob B of the pendulum is (with notation of above Article)

$$\sqrt{g(a^2-x^2)/l}, \text{ or } 2\pi\sqrt{a^2-x^2}/\mathrm{T},$$

if T is time of a complete vibration.

EXAMPLE 3. Show that the expressions for the velocity may be written in the forms

$$(1)\ \sqrt{gl(\alpha^2-\theta^2)};\ (2)\ 2\pi l\sqrt{\alpha^2-\theta^2}/\mathrm{T}.$$

EXAMPLE 4. Show that the velocity at the lowest point may be written in the forms

$$(1)\ 2\pi a/\mathrm{T};\ (2)\ a\sqrt{g/l};\ (3)\ \alpha\sqrt{gl}.$$

EXAMPLE 5. If the velocity at the lowest point is denoted by V, show that the velocity t units of time after passing the lowest point is

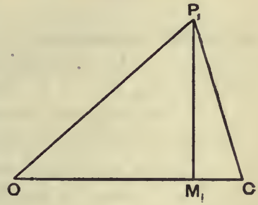
$$\mathrm{V}\cos(2\pi t/\mathrm{T}).$$

176. Propositions Regarding Moments of Inertia.

Prop. 1. If M be mass of any body, I the moment of inertia about any axis through the C.G., I′ that about a parallel axis at distance h from C.G., then

$$\mathrm{I}'=\mathrm{I}+\mathrm{M}h^2.$$

Break up the mass M into infinitesimal masses $m_1, m_2, m_3, \ldots$ at the points $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3, \ldots$ Let the plane of the paper be the plane through P_1, perpendicular to the parallel axes, O and C being respectively the points where this plane cuts the axis through the C.G. and the parallel axis. Let $\mathrm{P}_1\mathrm{M}_1$ be perpendicular to OC, and let $\mathrm{OM}_1=x_1$. Then, since $\mathrm{OC}=h$,



$$\mathrm{CP}_1{}^2=\mathrm{OP}_1{}^2+\mathrm{CO}^2-2\mathrm{CO}\,.\,\mathrm{OM}_1 \text{ (Euclid II. 12)}$$
$$=\mathrm{OP}_1{}^2+h^2-2hx_1;$$
$$\therefore\ m_1\mathrm{CP}_1{}^2=m_1\mathrm{OP}_1{}^2+m_1h^2-2h(m_1x_1).$$

Write down similar equations for each of the masses m_1, m_2,

m_3, . . . and add. Remembering the definition of moment of inertia, it is easily seen that on adding we shall obtain the equation

$$I' = I + h^2(m_1 + m_2 + m_3 + \ldots) - 2h(m_1x_1 + m_2x_2 + m_3x_3 + \ldots).$$

Now $m_1x_1 + m_2x_2 + m_3x_3 + \ldots$ is the sum of the products obtained by multiplying the mass of each element by its distance from the plane through O perpendicular to OC, and is therefore (see foot-note, page 254) equal to $M\bar{x}$, where $\bar{x}$ is the distance of the C.G. from the same plane. But $\bar{x} = 0$, because the C.G. lies in that plane, and therefore

$$m_1x_1 + m_2x_2 + m_3x_3 + \ldots = 0.$$

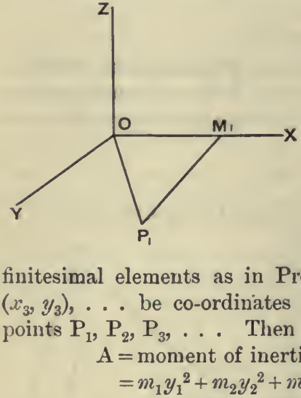
Also

$$m_1 + m_2 + m_3 + \ldots = M.$$

Hence the above formula for I′ becomes

$$I' = I + Mh^2.$$

Prop. II. If A and B denote respectively the moments of inertia of a *lamina* about two axes OX, OY at right angles, lying in the plane of the lamina, then C, the moment of inertia about an axis OZ through O perpendicular to plane of lamina, is given by the formula



$$C = A + B.$$

Break up the body into infinitesimal elements as in Prop. I., and let (x_1, y_1), (x_2, y_2), (x_3, y_3), . . . be co-ordinates of masses m_1, m_2, m_3, . . . at points P_1, P_2, P_3, . . . Then

$$\begin{aligned} A &= \text{moment of inertia about OX} \\ &= m_1y_1^2 + m_2y_2^2 + m_3y_3^2 + \ldots; \\ B &= \text{moment of inertia about OY} \\ &= m_1x_1^2 + m_2x_2^2 + m_3x_3^2 + \ldots; \end{aligned}$$

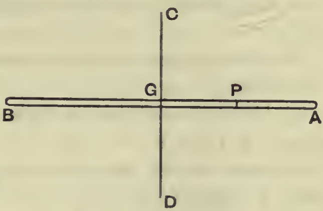
and

$$\begin{aligned} C &= m_1OP_1^2 + m_2OP_2^2 + m_3OP_3^2 + \ldots \\ &= m_1(x_1^2 + y_1^2) + m_2(x_2^2 + y_2^2) + m_3(x_3^2 + y_3^2) + \ldots \\ &= (m_1y_1^2 + m_2y_2^2 + m_3y_3^2 + \ldots) \\ &\quad + (m_1x_1^2 + m_2x_2^2 + m_3x_3^2 + \ldots); \end{aligned}$$

or

$$C = A + B.$$

Example 1. Find the moment of inertia of a uniform rod, AB, of length $2a$, about an axis CD perpendicular to its length through its middle point G. Break up each of the halves GA, GB of the rod into n equal parts; take the sum of the moments of inertia of the several parts about the axis CD; and in the expression for the sum make n infinitely great. (Compare Ex. 2, Art. 172.)



Let $a/n=d$, so that d is the length of each infinitesimal element of the rod. Take an element at P, where $\mathrm{GP}=xd$. If M be the whole mass of the rod, and m the mass of this element at P, we have

$$m/\mathrm{M}=d/2a,$$

and therefore $$m=\mathrm{M}d/2a.$$

The moment of inertia of this element about CD is

$$\begin{aligned}&=(\mathrm{M}d/2a).\mathrm{GP}^2=(\mathrm{M}d/2a).x^2d^2\\&=\mathrm{M}d^3x^2/2a.\end{aligned}$$

By putting $x=1$, $x=2$, $x=3$, . . . $x=n$, successively in this formula, and adding the results, we get the moment of inertia of the half GA of the rod about the axis CD. Similarly for the half GB. Hence the moment of inertia of the whole rod about CD

$$\begin{aligned}&=\Sigma 2\,\mathrm{M}d^3x^2/2a\\&=(2\,\mathrm{M}d^3/2a).(1^2+2^2+3^2+\ .\ .\ .\ +n^2)\\&=(\mathrm{M}d^3/a).\frac{n(n+1)(2n+1)}{6}\quad\text{(by Algebra)}\\&=(\mathrm{M}/6a)\,n^3d^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right).\end{aligned}$$

Now $nd=a$, and $1/n=0$ when n is infinite. Hence the moment of inertia of the rod about CD is

$$(\mathrm{M}/6a).a^3.2=\mathrm{M}a^2/3.$$

Example 2. Show (by Prop. I.) that the moment of inertia of the same rod about an axis through either end perpendicular to the length of the rod is $4\,\mathrm{M}a^2/3$.

Example 3. By breaking up a rectangle into strips parallel to a side, show that the moment of inertia of a rectangle, sides $2a$ and $2b$, mass M,

(1) About an axis in its plane through its centre perpendicular to sides $2a$, is $\mathrm{M}a^2/3$, and about an axis perpendicular to sides $2b$, is $\mathrm{M}b^2/3$.

(2) About an axis through its centre perpendicular to its plane, is $\mathrm{M}(a^2+b^2)/3$.

(3) About one of the sides $2a$, is $4\,\mathrm{M}b^2/3$; and about one of the sides $2b$, is $4\,\mathrm{M}a^2/3$.

(4) About an axis through an angular point perpendicular to its plane, is $4\,\mathrm{M}(a^2+b^2)/3$.

Example 4. The moment of inertia of a circle, radius a, mass M, about an axis through its centre perpendicular to its plane is $\mathrm{M}a^2/2$.

Break up the circle into rings of infinitesimal breadth by concentric circles.

Example 5. From the result of Ex. 4 show that the moment of inertia of a circle about a diameter is $\mathrm{M}a^2/4$.

Example 6. Show that the moment of inertia of a sphere, radius a, mass M, about a diameter, is $2\,\mathrm{M}a^2/5$.

Break up the sphere into circular laminæ by planes perpendicular to axis, and use Ex. 4.

Example 7. *Assuming* the formula for the length of the simple equivalent pendulum (Art. 101), *prove* that the centres of suspension and oscillation in the compound pendulum are convertible; that is, in notation of Art. 102, prove that

$$\mathrm{K}'^2/h' = \mathrm{K}^2/h.$$

Example 8. Find the length of the simple equivalent pendulum when a uniform rod, 12 feet long, swings about a horizontal axis perpendicular to its length (1) through one end of the rod, (2) through a point 2 feet from one end, (3) through a point 3 feet from one end.

Results.—(1) 8 feet; (2) 7 feet; (3) 7 feet.

Example 9. Find the length of the simple equivalent pendulum when a uniform rectangle, sides $2a$, $2b$, swings about a horizontal axis (1) coinciding with a side $2a$, (2) through an angular point perpendicular to the plane of the rectangle.

Results.—(1) $4b/3$; (2) $4\sqrt{a^2+b^2}/3$.

Example 10. Find the length of the simple equivalent pendulum when a uniform circle, radius a, swings about a horizontal axis in the plane of the circle at a distance $a/2$ from the centre.

Result.—a.

177. Solutions of Problems by Construction.

The method of graphical construction, explained in Art. 168, can be applied to give the solution in many problems in Dynamics. No new principles are required, as the method is simply an extension of the parallelogram construction. The following examples illustrate the method:—

Example 1.—Determine the true course and velocity of a steamer going due north by compass at 10 knots an hour through a 4-knot current setting south-east, and determine the alteration by compass in order that the steamer may make a true northerly course.

(London University, Intermediate B. Sc., 1887.)

Take a point O, and draw OA due north, making its length 10 units (say inches); draw OB, 4 inches south-east; complete the parallelogram OACB, having OA and OB as adjacent sides; and join OC. Then OC represents the actual velocity of the steamer. On measuring OC it will be found to be 7·7 inches; and therefore the true velocity of the steamer is 7·7 knots per hour in the direction OC determined by this construction. To find the direction in which the steamer must steer to make a true northerly course, describe a circle with centre O and radius equal to OA; produce BO to B′, making OB′ equal to OB; and through B′ draw B′D due north, meeting the circle just drawn in D. Then OD is the direction in which she must steer.

Example 2.—Forces of 7 and 16 units have a resultant of 21 units; find the directions of the forces by a construction drawn to scale.

(Sc. and Art Examination, 1st Stage, 1885.)

Construct a triangle ABC, by Euclid I. 22, having the sides AB, BC, CA respectively equal to 7, 16, and 21 inches: draw AD parallel to BC, and CD parallel to BA. Then AB, AD, AC represent respectively the forces 7, 16, and 21 in magnitude and direction.

Example 3.—Let D be middle point of BC, a side of the triangle ABC; join AD; let forces act from A to B, A to C, A to D, and from B to C, and let them be proportional to the lengths of those lines respectively; find their resultant by construction.

(Sc. and Art Examination, 2d Stage, 1885.)

The resultant of AB and AC is 2AD, and therefore the resultant of AB, AC and AD is 3AD. Hence the following construction:—Produce AD to E, making $DE=3AD$; produce DC to F, making $DF=2DC$; on DE and DF construct parallelogram DEGF. DG represents the resultant.

EXAMPLE 4.—A rod, AB, of uniform density is fastened by a hinge at A to the lowest point of a hemispherical bowl, and rests with the other end B against the inner surface of the bowl. Find the pressure on the bowl at B and on the hinge at A, the length of the rod being equal to the radius of the bowl.

(Sc. and Art Examination, 2d Stage, 1885.)

If O is the centre of the bowl, OAB is an equilateral triangle. Two of the forces acting on the rod, viz. the reaction at B and the weight, pass through C, the middle point of OB, and therefore the third force, the reaction of the hinge at A, also passes through C. Hence the right-angled triangle OAC has its sides OA, CO, CA parallel respectively to the forces, the weight, W, the reaction at B, R (say), and the reaction of the hinge, S (say). It easily follows that $R=W/2$, and $S=W\sqrt{3}/2$.

EXAMPLE 5.—A thread, 9 feet long, has its ends fastened to the ends of a rod, 6 feet long. The rod is supported in such a way as to be capable of turning freely round a point 2 feet from one end. A weight is placed on the thread—like a bead on a string. Find the position in which the rod will come to rest. It is supposed that the rod is without weight, and that there is no friction between the weight and the thread.

(Sc. and Art Examination, 2d Stage, 1876.)

Let ACB be rod, capable of turning round C, where $AC=2$ ft.; and let AOB represent in a figure the position of equilibrium, O being the position of the weight on the thread. By considering the equilibrium of the weight and the rod in succession, it is seen that OC must be vertical, and must bisect the angle AOB. Hence (Euclid vi. 3) OA is 3 feet, and OB is 6 feet; and it can be shown that OC is $\sqrt{10}$ feet. Hence construction:—Measure CO, $\sqrt{10}$, vertically downwards; and construct the triangles OCA and OCB, whose sides are known. Then ACB will be a straight line, and the position of equilibrium will be completely determined.

EXAMPLE 6.—ABC is a triangle right-angled at C; its sides are uniform rigid rods AB, BC, CA loosely jointed together; AB is heavy, but the others are so light that they can be treated as weightless; the whole hangs from the point C, and can turn freely round it. Find the position of equilibrium, and the forces acting on the rods severally.

(Sc. and Art Examination, 2d Stage, 1882.)

The only forces acting on BC are the reactions at B and C; these must therefore act along BC, and must be equal and opposite. Similarly for CA. If R and S denote respectively the reactions along BC and CA, and W the weight of AB, we shall obtain by the triangle of forces

$$R = W . BC/AB \; ; \; S = W . CA/AB.$$

Hence, knowing W, we can determine R and S by measuring the lengths of BC, CA, AB. Also the vertical through C must pass through G, the middle point of AB. Hence the angle ACG is equal to the angle CAG, which determines the position of equilibrium.

EXAMINATION PAPERS.

I.

THEORETICAL MECHANICS.

First Stage.

[You are not permitted to *attempt* more than *seven questions.* You may select them from any part of the paper.]

1. State how to find the position of the centre of gravity in the following cases:—(*a*) a square lamina, (*b*) a triangular lamina, (*c*) a cone; each body being of uniform density.

2. Forces of 10, 13, and 16 units act at a point and are in equilibrium. Show by a diagram how they act.

3. A rod AB can turn freely round a hinge at A, and rests in an inclined position with its end B against a smooth vertical wall. Show in a diagram how the forces act which keep the rod at rest, and name them.

4. Define the moment of a force with reference to an assigned point.

A uniform rod rests in a horizontal position on two supports 8 ft. apart, one under each end; it weighs 6 lbs.; a weight of 24 lbs. is hung to it from a point distant 3 ft. from one of the points of support. Find the pressure on each support.

5. State what is meant by the sensibility of a balance, and explain why the sensibility is great when the arms are long, and also when the centre of gravity of the beam is near the point of support.

6. Define a foot-pound of work.

A man weighing 140 lbs. puts a load of 100 lbs. on his back, and carries it up a ladder to a height of 50 ft. How many foot-pounds of work does he do altogether, and what part of his work is done usefully?

7. If the velocity of a body is increased uniformly in each second by 32 feet a second, by how many feet a second is its velocity increased in one minute?

If the velocity is 1920 ft. a second, what is it in yards a minute?

8. Write down the formula for the distance described in a given time by a body whose velocity is uniformly accelerated.

A body thrown upwards against gravity reaches a greatest height of 121 ft. Find the velocity with which it is thrown up, and the number of seconds that will elapse before it returns to the point of projection. ($g=32.$)

9. A body whose mass is 10 lbs. is moving at the rate of 10 ft. a second. What is the numerical value of its kinetic energy at that instant?

If from that instant it moves against a constant resistance equal to one-twentieth of its weight, how far does it go before being brought to rest? ($g=32.$)

10. When a body is wholly or partly immersed in a liquid what is the magnitude of the resultant pressure of the liquid on the body?

A body, whose specific gravity is 1·4 and volume 3 cubic ft., is placed in a vessel in which there is water enough to cover it. What pressure does the body produce on the points of the bottom of the vessel at which it is supported?

11. Explain briefly the method of finding the specific gravity of an insoluble body by means of the balance.

If the body weighs 732 grains in vacuo and 252 grains in water, what is its specific gravity?

12. A tube filled with water is inverted with its open end in water, no air having got in; the top of the tube is 20 ft. above the surface of the external water. If the water barometer stands at 34 ft., what is the pressure in pounds per square foot at a point of the inside of the top of the tube? (Cubic foot of water weighs 1000 oz.)

What would be the consequence of making a hole through the top of the tube, and why would this consequence follow?

II.

SECOND STAGE.

[You are not permitted to *attempt* more than *eight questions.* You may select them from any part of the paper.]

1. State and prove the rule for finding the resultant of two parallel forces acting towards the same part of a rigid body.

Parallel forces of 10 and 12 units act towards the same part at

A and B; a force of 15 units acts from A to B. Find resultant of the three forces, and show in a diagram how it acts.

2. Show that two couples whose moments are equal and of opposite signs, are in equilibrium when they act in the same plane on a rigid body.

3. ABCD is a quadrilateral figure; P and Q are the middle points of the opposite sides AB and CD; O is the middle point of PQ. Show that four forces represented by OA, OB, OC, OD respectively are in equilibrium.

4. ABC is a rigid equilateral triangle (whose weight is put out of the question); the vertex B is fastened by a hinge to a wall, while the vertex C rests against the wall under B. If a given weight is hung from A, what are the magnitude and direction of the forces exerted by the triangle on the wall at B and C?

5. Define the coefficient of friction.

A weight of 500 lbs. is placed on a table and is just not made to slide by a horizontal pull of 155 lbs. Find the angle of friction by drawing it to scale; or, if you have no instruments, explain how to calculate the number of degrees.

6. Find the relation between the power and the weight in the screw, taking into account the friction between the threads of the screw and of the companion screw.

7. Define a foot-pound of work and a horse-power.

A steam crane working with 3 horse-power is found to raise a weight of 10 tons to a height of 50 ft. in 20 minutes. What part of the work is done against friction? If the crane is kept at similar work for 8 hours, how many foot-pounds of work are wasted on friction?

8. Find the position of a body at the end of a given time from the instant at which it is thrown up with a given velocity in a given direction, the motion being supposed to take place in vacuo.

A body is thrown up in a direction making an angle of 30° with the horizon, and passes through a point whose horizontal distance from the point of projection is $400\sqrt{3}$ feet, and vertical height above the point of projection 76 feet. Find the velocity of projection. ($g=32$.)

9. A particle whose mass is 10, moving with a velocity 5, meets and impinges directly on another particle whose mass is 20 and velocity 3; the coefficient of restitution being ·125. Find from first principles the velocities of the particles at the end of the impact.

State the dynamical principles employed in answering this question, and define coefficient of restitution.

10. A fly-wheel weighs 10,000 lbs., and is of such a size that the matter comprising it may be treated as if concentrated on the circumference of a circle 12 feet in radius. What is its kinetic energy when moving at the rate of 15 revolutions a minute? How many turns would it make before coming to rest if the steam were cut off, and it moved against a friction of 400 lbs. exerted on the circumference of an axle of 1 foot in diameter? ($g=32$.)

11. Define the centre of pressure of a fluid on a plane area, and find its position in the case of a rectangular area, one edge being in the surface of the fluid.

Find where the centre of pressure of the rectangle would be if its plane were vertical and its upper edge (which is horizontal) below the surface at a distance equal to the height of the rectangle.

12. Given the specific gravities of two liquids, show how to calculate the specific gravity of a mixture of given volumes of the two liquids, assuming that the mixture takes place without change of volume.

3 pints of a liquid, whose sp. gr. is ·8, are mixed with 5 pints of another liquid, whose sp. gr. is 1·04; find the sp. gr. of the mixture (*a*) if there is no contraction, (*b*) if on mixture there is a contraction of 5 per cent of the joint volumes.

III.—UNIVERSITY OF LONDON.

MATRICULATION EXAMINATION—NATURAL PHILOSOPHY.

[Only *eight* questions are to be answered, of which *at least two* must be taken from Section A.]

A.

1. Define velocity, acceleration, force, momentum. What is meant by uniformly accelerated motion? A heavy body slides down a smooth inclined plane, inclined at 30° to the horizon. Through how many feet will it fall in the fourth second of its motion? Take $g=32$ in foot-second units.

2. Two heavy particles are connected by a flexible string which passes over a fixed pulley. Show how to find the acceleration with which the heavier body will descend.

If the masses are 17 oz. and 15 oz., find the tension of the string.

3. What is meant by the *mechanical advantage* of a system of pulleys or other machine?

Find relation between power P and weight W in the system of 5 movable pulleys, in which each pulley hangs by a separate string and the weight of each pulley is equal to P.

4. What is meant by the centre of gravity of a body? Show how to find the C.G. of a system of heavy particles lying in one plane. Weights of 1 lb., 2 lbs., 3 lbs., 4 lbs., are placed at the angular points A, B, C, D of a square ABCD. Find the distance of the C.G. of the system from the centre of the square.

5. Distinguish between *weight* and *mass*.

A cannon ball whose mass is 60 lbs. falls through a vertical height of 40 feet? What is its energy?

With what velocity must such a cannon ball be projected from a cannon to have initially an equal amount of energy?

B.

6. Distinguish between a solid and a fluid. What is the special characteristic of a perfect fluid?

A cube of one foot edge is suspended in water with its upper face horizontal, and at a depth of 2½ feet below the surface. Find the pressure on each face of the cube, assuming that the mass of a cubic foot of water is 1000 oz.

7. Define specific gravity, and explain the principle of the common hydrometer. The stem of a common hydrometer is cylindrical, and the highest graduation corresponds to a sp. gr. of 1, while the lowest corresponds to a sp. gr. of 1·2. What sp. gr. corresponds to the point which is exactly midway between these two divisions?

8. What conditions limit the height to which water can be raised by means of a common pump? How must the pump be worked when it is required to raise water to a height greater than the limit thus assigned? Explain the contrivance in the lift-pump for enabling the piston-rod to work water-tight through the cylinder cover.

9. How would you determine the sp. gr. of a gold medal by means of a hydrostatic balance furnished with brass weights?

Explain how each weighing and the final result would be affected by the presence of the air if no correction is made for the air displaced.

This paper also contains questions on Physics.

IV.—UNIVERSITY OF EDINBURGH.

MEDICAL PRELIMINARY EXAMINATION—MECHANICS.

1. Define velocity, and show how it is measured. Give an example of a simple body in motion, the different points of which have different velocities.

2. Show how to resolve a velocity in two directions at right angles to each other. If a point has a velocity of 36 yards per minute in a direction making an angle of 30° with the vertical, find its horizontal component.

3. Investigate the expression

$$v^2 = V^2 + 2as,$$

for the velocity of a point moving under the action of a uniform acceleration and in the direction of motion, where V is the initial velocity and s the space described. What is the physical interpretation of this equation?

4. A particle whose initial velocity is 6 feet per second moves for 5 seconds under a uniform acceleration, and acquires a velocity of 21. Find the acceleration, average velocity, and the space described.

5. A weight of 3 lbs., resting on a smooth table, is attached by a light string to another weight of 1 lb. which hangs over the edge of the table. If the system starts from rest, find the space described in 4 seconds, supposing the table so large that the first weight does not leave it. If it just leaves it at the end of the fourth second, how do you find its subsequent motion?

6. Define the centre of mass of a body, and give a method of finding it experimentally.

7. Two weights of 3 and 5 lbs. are hung on one side of a lever, at distances of 7 and 9 inches respectively from the fulcrum. What weight must be hung on the other side of the fulcrum, 3 inches from the fulcrum, so as to balance these?

8. A uniform heavy rod AB rests on the ground at B, and against a smooth vertical wall at A. If a stretched string without weight join C, the middle point of AB to D, the foot of the perpendicular from B on the wall, give a diagram showing the forces which act on AB. If the ground is smooth, could AB remain in equilibrium? Give a reason for your answer.

9. Define the pressure at any point of a fluid. Prove that if no external forces act, the pressure is the same at all points and in all directions. If external forces act, what does the proposition become?

10. Find the pressure at a point 20 feet below the surface of still water freely exposed to the atmosphere, given that a cubic foot of water weighs 1000 ounces.

V.—UNIVERSITY OF EDINBURGH.

M.A. PASS EXAMINATION—NATURAL PHILOSOPHY.

1. Distinguish between *uniform* and *variable* acceleration.

Give examples of bodies having accelerations:—

- (*a*) Constant in magnitude and direction.
- (*b*) Constant in magnitude but not in direction.
- (*c*) Constant in direction but not in magnitude.
- (*d*) Variable both in magnitude and direction.

2. With uniform acceleration in the direction of motion show that a point describes spaces proportional to the *squares* of the times elapsed since the commencement of the motion.

3. Distinguish between *Energy* and *Work*, and explain what is meant by the *Conservation of Energy.*

A simple pendulum is pulled aside till its heavy bob is raised m inches vertically and then let go. Find its velocity when it passes its lowest point.

4. P and Q are two parallel forces. Find the magnitude of their resultant and its line of action when the forces are (1) like, (2) unlike.

Discuss the case in which the forces are unlike and equal.

5. Define the *Centre of Mass* (*Inertia*), and state without proof its position in the following cases:—

- (*a*) A uniform triangular lamina.
- (*b*) Three equal heavy particles at the angular points of a triangle.
- (*c*) The perimeter of a triangle formed by a piece of uniform wire.

A and B are two masses m and n, G is their centre of mass, and P is any other point. Show that

$$m \cdot \mathrm{AP}^2 + n \cdot \mathrm{BP}^2 = m \cdot \mathrm{AG}^2 + n \cdot \mathrm{BG}^2 + (m+n)\mathrm{PG}^2.$$

6. State the elements which specify a force.

What is meant by an *absolute unit* of force, and what fundamental units does it involve?

7. Describe the *essential parts* of Attwood's machine, and state to what use it is applied.

8. Prove that when a body moves with uniform acceleration in a fixed direction its hodograph is a straight line.

What is the hodograph in the case of (1) a projectile, (2) a planet?

9. What is meant by the *Centre of Pressure?*

When does the Centre of Pressure coincide with the Centre of Inertia?

A rectangle is immersed in water with one side in the surface. Show how to divide it by two horizontal lines into three parts, on each of which the whole pressure shall be the same.

10. Explain how to find the specific gravity of a solid body that floats in water.

11. Two liquids that do not mix are contained in a bent tube; the difference of their levels is p inches, and the height of the denser above their common surface is q inches. Compare their specific gravities.

This paper also contains Questions on Physics.

VI.—UNIVERSITY OF GLASGOW.

M.A. PASS EXAMINATION—NATURAL PHILOSOPHY.

FIRST PAPER.

1. Define force, density, velocity, momentum.

In what systems of units are these quantities generally measured?

2. Enunciate the Parallelogram of Forces.

Assuming it true for the direction of the resultant, show that it gives correctly the magnitude of the resultant.

How may the theorem be illustrated experimentally?

3. At what angle must two forces of 6 and 10 dynes respectively be placed to have a resultant of 14 dynes?

If the angle is doubled, what does the resultant become?

4. Find the resultant of two like parallel forces.

A rod, 4 feet long, has weights hung from the ends and from each of the foot divisions. The weights taken in order are 8, 4, 7, 10, and 5 lbs. Find where the rod must be supported to balance.

5. Define the centre of gravity of a triangle, and show that it coincides with the centre of gravity of three equal particles placed at the angular points.

6. Sketch the system of pulleys in which each string is fastened to the weight.

Is it more advantageous in raising a weight by this system to have the pulleys light or heavy?

7. Enunciate Newton's laws of motion.

Show that the second law gives us the means of measuring force and also of measuring the mass of a body.

A string connecting two weights of 5 and 7 lbs. is hung over a pulley. Find the acceleration of either weight.

8. A particle moves with acceleration f during a time t through a space s. Its initial and final velocities are u and v. Prove that

$$v^2 = u^2 + 2fs.$$

If the mass of the particle is m, find the kinetic energy of the particle at time $\frac{1}{2}t$ in terms of u, v, m.

9. A train moving at the rate of 40 miles an hour is brought to rest by sliding friction in half a minute. Show that the coefficient of friction is $\frac{11}{180}$.

10. Find the acceleration of a particle moving uniformly in a circle.

A stone, weighing 1 lb., is swung round at the end of a string 3 feet long in a horizontal circle, making 2 revolutions per second. Find its tension.

11. How may the principle of the Conservation of Energy be stated in its application to dynamics?

Show its connection with the axiom that "the Perpetual Motion" is impossible.

How many foot-pounds of work are done by friction in stopping the train in Question 9?

VII.

SECOND PAPER.

1. State Archimedes' principle in hydrostatics. Explain the method of finding the specific gravity of a solid lighter than water, and investigate a formula corresponding to your description of the method for finding the specific gravity of the body.

2. A rectangular board, 3 feet by 2 feet, is immersed in water with its long edges horizontal. The upper of these edges is 7 feet below the surface, and the lower 8 feet 6 inches below the surface. Find the pressure on the board.

3. Describe Nicholson's Hydrometer, and explain how it is used (1) for finding the specific gravity of a solid, (2) for finding the specific gravity of a liquid.

4. Define "height of the homogeneous atmosphere." Show that it is independent of the barometric height, and explain how it is affected by temperature.

Calculate the height of the homogeneous atmosphere at 0° C. from the following:—

Mercury is 13·596 times as dense as water, and the weight of 1 cubic centimetre of air at 0° C. and 760 millimetres pressure is ·001293 gramme.

This paper also contains Questions on Physics.

VIII.—WOOLWICH ROYAL MILITARY ACADEMY.

FURTHER EXAMINATION—STATICS.

[Great importance will be attached to accuracy.]

1. Two forces are given in magnitude, but may make any angle with each other. How should they be placed so as to give (1) the greatest, (2) the least possible resultant?

2. What is meant by a force resolved in a given direction?

Show how a force is to be resolved so that its component along a given direction shall have a given value.

3. A force is given in magnitude and line of action. Give a geometrical construction for resolving it into two other forces which shall be equal to one another and shall pass through two fixed points.

4. Prove that any two couples of equal moment and opposite senses balance each other.

(It may be assumed that the same couple can be transferred in its own plane without changing its effect.)

5. Three parallel forces act on a horizontal bar, each being $=1$ lb.; the right hand force acts vertically upwards, the other two vertically downwards, at distances of 2 feet and 3 feet respectively from the first. Draw their resultant, and state exactly its magnitude and position.

6. Draw any system of pulleys by which a weight of 1 lb. can be made to support a weight of 3 lbs., neglecting friction and the weight of the pulleys. Show that whatever may be the system, the smaller weight will descend 3 feet in raising the other through 1 foot.

7. A heavy circular disc is kept at rest on a rough inclined plane by a string parallel to the plane and touching the circle. Show that the disc will slip on the plane if the coefficient of friction is less than $\frac{1}{2}\tan i$, where $i=$ angle of slope.

8. Two equal weights, each $=112$ lbs., are joined by a string which is laid over two pulleys A and B in the same horizontal line. If a small weight, say 1 lb., is attached to the string half-way between A and B, find in inches the depth to which it descends below the level of AB, supposing $AB=10$ feet.

What would happen if the weight were attached to any other part of the string?

9. If a portion m of any mass M is moved to any new position, show that the centre of gravity of the entire mass is thereby moved in a direction parallel to the displacement of the centre of gravity of m and over a distance

$$\frac{m}{M}D,$$

where $D=$ above distance between the two positions of the centre of gravity of m.

A triangular piece of paper is folded across the line bisecting two sides, the vertex being thus brought to lie in the base. Find the C.G. of the paper in this position.

10. Three equal particles are placed anywhere on the three sides of a triangle. If they are moved along those sides in the same sense and over spaces respectively proportional to the sides, show that the C.G. of the particles remains at rest.

11. Assuming the principle of virtual velocities, deduce the relation between the power and the weight in the inclined plane, the power being either (1) parallel to the plane or (2) horizontal.

12. Explain what is meant by stable and unstable equilibrium, and give an instance of each.

IX.

DYNAMICS.

[The measure of the acceleration of gravity may be taken to be 32 in foot-second units. Great importance will be attached to accuracy.]

1. State the theorem known as the parallelogram of velocities.

A stream runs with the velocity of 1½ miles per hour. Find in what direction a swimmer, whose velocity is 2¼ miles per hour, should start in order to cross the stream perpendicularly.

2. A person walking along a road at rate v sees a tower a mile distant from his eye. The nearest distance of the tower from the road is ½ mile. Find the rate at which he is approaching the tower. Does this rate alter as he advances?

3. A sphere at rest is struck directly by another in motion. Describe in a few words the essential points of the phenomenon of impact—(1) when the spheres have no elasticity, (2) when they are perfectly elastic.

4. A particle is shot vertically upwards with velocity u. Find the time of ascent, and show that it is equal to the time of descent.

5. If a number of particles are let fall at the same instant down a number of smooth inclined planes having a common vertex, prove that at any moment they all lie on one circle.

Prove also that if all the planes are equally rough, the particles will be on one circle.

6. A ball is discharged with the initial velocity of 1100 feet per second. Find in miles the greatest range it can give on a horizontal plane.

Could this range be reached in practice?

7. A projectile is thrown with given velocity and elevation. Give the horizontal and vertical components of its velocity at a given time. If, after a given time, a second projectile is thrown from the

same point in the same trajectory, describe what appearance its motion would present as seen from the first.

8. One end of a string is fixed; it then passes over a movable pulley to which a weight W is attached. The string then passes over a fixed pulley, and a smaller weight P is attached to its other end, all three sections of the string being vertical. Show that, neglecting the masses of the pulleys, the acceleration with which W descends is

$$\frac{W-2P}{W+4P}g.$$

Verify this result when P is small compared with W, and when W is small compared with P.

9. A bead is strung on a thread whose two extremities are fixed, and on which it can move without friction. Show that if the bead is started so as to move in the elliptic arc to which it is constrained, its velocity will be uniform.

Show also that the tension of the thread for different positions of the bead varies inversely as the product of the two focal radii vectores.

10. Define the work done by a force when it continues to act while its point of application moves over a given space in the direction of the force. If that point moves at an angle with the force what is the work done?

A mass m is moving with velocity u. A constant force acts on it in the direction of the motion until its velocity is v. Prove that the work done by the force is

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mu^2.$$

X.

THEORETICAL MECHANICS—SOLIDS.

SECOND STAGE OR ADVANCED EXAMINATION.

(*Answers on page 304.*)

[You are not permitted to attempt more than *eight* questions.]

1. Define the angular velocity of a moving point with respect to a fixed point. Under what circumstances will the angular velocity

of the moving point be equal to its linear velocity divided by its distance?

Draw an equilateral triangle ABC, having each side 12 feet long; a point moves along BC with a velocity of 10 feet a second; when it is at C, what is its angular velocity with respect to A?

2. State Newton's Second Law of Motion. Explain briefly how the measure of force is derived from this Law.

In the equation $P = mf$, in what units is P, when the units of mass, distance, and time are a pound, a foot, and a second?

3. Define a couple. Explain how to find the resultant of two forces which form a couple and a third force.

Draw a square ABCD; a force of 8 units acts from A to D, and two parallel forces of 12 units act from A to B and C to D respectively; find the resultant. Also find what the resultant would be if the first force acted from D to A.

4. State and prove the rule for finding the centre of gravity of a triangular pyramid of uniform density.

5. When two smooth bodies are pressed together, in what direction does the mutual action take place? If the bodies are rough, what other force may be called into play?

A particle of given weight is placed on an inclined plane and stays at rest; what is the magnitude of the friction called into play? Under what circumstances would the particle stay at rest if the inclination of the plane were increased?

6. A uniform rod rests in limiting equilibrium against a smooth fixed point C, with one end on a rough horizontal plane; given the height of C above the plane, and the inclination of the rod to the horizon, find the length of the rod.

For a given coefficient of friction what is the steepest inclination of the rod?

7. State the principle of virtual work (or virtual velocities).

Verify the principle in the case of the first system of pulleys, when there are three movable pulleys (each string being attached to the supporting beam).

8. A curve is drawn, and AN, NP are the abscissa and ordinate of any point (P) of the curve; if AN represents the distance through which a force has acted, and PN represents the magnitude of the force when it has acted through that distance, show that the area of the curve represents the work done by the force.

9. Two circles touch each other externally, and the point of contact (A) is in the same vertical line as the centres; from any point (P) of the upper circumference draw a straight line PAQ to meet the lower circumference in Q; if a particle is allowed to fall from P along PQ, show that the time it takes to reach Q is constant for all positions of P.

Also compare the times in which PA and AQ are described.

10. A particle slides down a rough inclined plane, find the acceleration of its velocity.

Under what circumstances would the velocity be retarded?

The angle of friction between a particle and a plane is 30°, and the angle of inclination is 15°; if the particle begins to slide down with a velocity of 100 feet a second, how far will it slide before coming to rest?

11. A particle of given mass moves with a given velocity in a circle of given radius; state what is known as to the force which acts on the particle.

Prove the statement.

12. A particle (A), whose mass is 3, is tied by an inextensible thread to a particle (B), whose mass is 5, and B is placed on a smooth table, while A is allowed to fall from the edge; at first the thread is slack, but at the instant A has fallen 9 feet, the thread is drawn tight; with what velocity does B begin to move?

What part of the kinetic energy of the system disappears when the thread is drawn tight?

XI.—UNIVERSITY OF LONDON.

INTERMEDIATE EXAMINATION IN SCIENCE.

Mixed Mathematics.

(*Answers on page 305.*)

1. Two forces act at a point P along the straight lines PA, PB, where A, B are fixed points; the magnitudes of the forces are represented by $m.PA$, $n.PB$ respectively, where m, n are given numbers. Prove that, for all positions of P, the resultant of the forces passes through a fixed point on the straight line AB, and that its magnitude is represented by $m+n$ times the distance of P from that point.

2. Two wires are fixed at right angles to one another, and are each inclined at an angle of 45° to the horizon, the wires meeting at their highest point.

A light string has small smooth rings, each of mass m, fastened to its ends, and these rings slide one on each wire; at the middle point of the string a heavy bead, of mass M, is fastened, and rests in equilibrium.

Prove that the inclination θ of each half of the string to the vertical is given by

$$\tan\theta = 1 + \frac{2m}{M},$$

and find the pressure on each wire.

3. Find the magnitude and the line of action of the resultant of two given forces which act along parallel straight lines in opposite directions.

Like parallel forces act at the angular points of a triangle, and are proportional in magnitude to the opposite sides of the triangle; prove that their resultant passes through the centre of the circle inscribed in the triangle.

4. Having given the centre of gravity of a body of given mass, and also that of a part of known mass which is cut off, find the centre of gravity of the remaining part.

A piece of cardboard is in the shape of a square; one corner is bent over until it reaches the centre of the square, the crease being the line joining the middle points of the adjacent sides of the square. Find the centre of gravity of the cardboard so folded, a side of the square being a inches.

5. A common steel-yard is 3 feet long, the weight of the bar and scale-pan is 2 lbs., and its centre of gravity is 4 inches, and the fulcrum 3 inches from the end from which the scale-pan is suspended; the movable weight being 1 lb., show how to graduate the steel-yard, and find the greatest weight which it can be used to measure.

6. Explain how the velocity of a moving point relative to another moving point is measured.

A railway train is travelling at the rate of 30 miles an hour along a straight track; a bullet moving in a straight line perpendicular to the direction of motion of the train enters a carriage through a window and passes out through the opposite window at a point 3 inches from the point opposite that at which it entered; the breadth of the carriage being 10 feet, find the velocity of the bullet.

7. A stone is let fall from the top of a tower 50 feet high, and at the same instant a stone is thrown vertically upwards with such a velocity that the two stones may pass each other 9 feet from the top of the tower; with what velocity must the second stone be thrown, and what time will elapse before it again reaches the ground? The acceleration due to gravity may be taken as 32 feet per second per second.

8. Explain how a measure of force is obtained from Newton's second law of motion. Define a poundal, and find roughly the weight to which it is equivalent.

9. Having given that a poundal is equal to 13,825 dynes, a dyne being the unit of force in the C. G. S. system of units, and that a foot is ·304797 metres; find the number of lbs. in a gramme.

10. A mass m is on a smooth inclined plane of inclination α, and is fastened to a light string which passes without friction over the top of the plane, so that the string is along a line of greatest slope; at the other end of the string is fastened a particle of mass m', which hangs vertically. Find the acceleration with which m moves up or down the plane, and the tension of the string.

XII.

THEORETICAL MECHANICS—SOLIDS.

ADVANCED STAGE.

(*Answers on page* 306.)

[You are not permitted to attempt more than *eight* questions.]

1. Define the angular velocity of a moving point with respect to a given fixed point. Under what circumstances will a moving point have no angular velocity with reference to a given fixed point?

AB is a given straight line, and P a given fixed point without it; a particle Q moves along AB with a given constant velocity; when Q is in any assigned position, find its angular velocity with respect to P.

2. Two particles move in a straight line, and are acted on by forces P and Q respectively; the mass of the one particle is m, and its velocity is increased by f in a certain time; the mass of the other is m_1, and its velocity is increased in the same time by f_1; show that it follows from Newton's law that

$$P : Q :: mf : m_1 f_1.$$

Explain how we can deduce from this proportion the number of absolute units of force in P.

3. Three forces, P, Q, R, act at a point O, and are in equilibrium; show that

$$(a)\quad \frac{P}{\text{Sin QOR}} = \frac{Q}{\text{Sin ROP}} = \frac{R}{\text{Sin POQ}};$$

$$(b)\quad P^2 = Q^2 + R^2 + 2QR\ \text{Cos QOR}.$$

If the magnitudes of the three forces in equilibrium are 12, 15, and 19 respectively, show in a diagram drawn to scale how they act.

4. Define the centre of a system of parallel forces.

Given two parallel forces and the co-ordinates of the points at which they act, show how to find the co-ordinates of their centre.

There are two unlike parallel forces of 2 and 7 units; they act at points whose co-ordinates are $(-2, 1)$ and $(3, -4)$ respectively; find the co-ordinates of their centre, and show the two points and the centre in a diagram.

5. Draw a triangle ABC with AB vertical, and let AC and BC represent two weightless rods, joined together by a smooth hinge at C and fastened by smooth hinges to fixed points at A and B; a weight W is hung from C; show that one of the bars is in a state of tension, and the other of compression; also show how to calculate the stresses.

Obtain numerical results in the following case:—

$$AB=4,\ BC=3,\ CA=2, \text{ and } W=18 \text{ tons.}$$

6. Let μ denote the coefficient of friction, and R the normal reaction of a rough plane against a body which it partly supports; explain under what circumstances the friction between the body and the plane will equal μR; also explain whether the friction under any circumstances could be less than μR.

A uniform rod rests with one end against a smooth vertical wall, and the other end on a rough horizontal plane; it can just stand without sliding when its inclination to the horizon is 45°; find the coefficient of friction; also find the inclination when the friction called into play is one-half of the limiting friction.

7. Assuming the principle of virtual work, find the relation between the power and the weight in the case of a screw working without friction.

The pitch of a screw is $\frac{2}{3}$ of an inch, and the length of the arm is 30 inches; it is found that by means of the screw a weight of 3 tons can be raised by a force equal to the weight of 42 lbs.; find how many foot-pounds of work are done against friction when the weight is lifted a foot and a half. ($\pi = \frac{22}{7}$.)

8. A uniform rope hangs by one end, and carries a weight at the other; show how to draw a diagram to represent the work done in winding up the rope, and thereby lifting the weight.

9. A particle is projected in a given direction *in vacuo*; show that it describes a parabola.

A is the highest point, and S the focus of the path of a projectile; N is the point in which AS produced meets the horizontal plane through the point of projection; given that SN is twice AS, find the point of projection and the direction of projection.

10. A uniform and perfectly flexible thread (or chain) is placed on a smooth horizontal table so that one end just hangs over the

edge, and consequently that end falls, dragging the rest of the thread after it. Find (*a*) the acceleration of the velocity at the instant when an assigned length is hanging; (*b*) the work that has been done by gravity up to that instant; (*c*) the velocity of the thread at that instant.

State the mechanical principles that justify your results.

11. A particle falls from a given height on to a smooth horizontal plane; the coefficient of restitution between the particle and the plane is given; find the height of the first, and also of the second rebound.

12. Draw AB inclined at a given angle to the vertical and BC horizontal in such a way that A is above the prolongation of CB. Let AB represent a thread fastened to a fixed point at A, and to a heavy particle at B; the particle is kept in position by a force pulling it along a thread BC; after a time the thread BC breaks; show that the weight of the particle is a mean proportional between the tension of AB before BC breaks, and its tension immediately after BC breaks.

XIII.—UNIVERSITY OF LONDON.

INTERMEDIATE EXAMINATION IN SCIENCE. MIXED MATHEMATICS.

(*Numerical results are to be correct to two places of decimals.*)

[*Answers on page* 307.]

1. A weight of 100 lbs. is suspended from C, being supported by two rods, AC, BC, 6 and 8 feet long, freely jointed together, whose other ends A and B rest on a frictionless horizontal plane, and are kept from slipping by a cord AB, 10 feet long, connecting them; the plane ACB is vertical. Find the thrust in each rod and the tension in the cord arising from this load; also the fractions of the load sustained by the plane at A and B respectively.

2. Forces act upon a solid body, represented in position and magnitude by AB, twice BC, CA. Find the position and magnitude of their resultant.

Find the resultant of forces represented by AB, BD, DC, CA, where ABCD is a parallelogram of which AC, BD are the diagonals.

3. Prove that the centre of gravity of a uniform triangular plate ABC is the same as that of a frame composed of three uniform bars AB, BC, CA, of equal weights.

Also find what loads must be placed at the corners of a triangular plate whose sides are 3, 4, and 5 feet, in order that it may balance on the centre of the inscribed circle, the sum of the loads being 20 lbs., and the weight of the plate being neglected in comparison with them.

4. A light thread passes over two pulleys, A, B, one foot apart, at the same level, and carries a weight of 3 lbs. suspended from each end, and a weight of 2 lbs. suspended from its middle point. Find the depth below the level of AB at which its middle point will rest.

If the weights suspended are P, P, R, and the pulleys are not at the same level, show that the position of the point of attachment of the middle weight to the thread is given by the following construction: describe on AB an arc of a vertical circle containing a certain given angle; from the middle point of the complementary arc draw a vertical line, meeting the circle again in D; then D is the point of attachment.

5. Two balls, each weighing 10 lbs., are attached to the ends of two uniform rods, AB, AC, each 2 feet long and 4 lbs. weight, which are welded together at A at an angle of 60°; another rod, AO, 1 foot long and 2 lbs. weight, is welded to them so as to bisect the angle between them. Show that the system will balance on the lower end O of this rod as a pivot, so that A shall be above O; and find what force applied horizontally at A would be required to push that point sideways through 2 inches.

6. Prove, graphically or otherwise, that in uniformly accelerated motion the space described in any interval is equal to the time multiplied by the mean of the initial and final velocities.

A body under the action of a constant force is observed to move 20 feet in a certain second of time, and 25 feet in the following second. How far will it move in the minute next succeeding?

7. The acceleration due to gravity is 32·2 when the units are a foot and a second. What will be its value when the units are a kilometre and an hour, a metre being 39·37 inches?

8. A mass of 1 lb., moving along a frictionless horizontal plane at 10 feet a second, impinges directly upon a mass of 100 lbs. resting on the plane. What velocity will it impart to the latter, on the

supposition that the masses rebound with the same relative velocity as they approached?

9. If the weight on one side of an Attwood pulley is 10 lbs., find what must be the counterbalancing weight in order that this one may descend with one-tenth of the acceleration of a body falling freely, friction and the mass of the pulley being neglected.

A light thread passes over two fixed pulleys, A and B, and carries between them a movable pulley-block C, under which it passes; a mass M is attached to each end of the thread, and a mass m to the movable block; the masses of the pulley are negligible, and the pulleys are so arranged that all the segments of the thread are vertical. Show that when the system is let go the tension in the thread is $m\,\mathrm{M}/(\mathrm{M} + \frac{1}{2}m)$ lbs., and find the acceleration with which the mass m falls.

10. The engine of a train of 200 tons exerts a steady hauling force of 3000 lbs., and the frictional resistance to the motion of the train is 10 lbs. per ton. Find the time that the train would take to travel 5 miles starting from rest, (i) when the line is level, (ii) when there is a down gradient of 1 in 150 for the first 2 miles, and an up gradient of 1 in 450 for the remaining 3 miles.

ANSWERS TO EXAMPLES.

I. Page 13.

1. (1) 9. (2) $\frac{37}{1440}$. (3) $586\frac{2}{3}$. (4) 18. (5) $20\frac{1}{6}$. (6) $\frac{17}{450}$. (7) 88. (8) $\frac{11}{39}$. (9) $3a$. (10) $\frac{a}{12b}$. (11) $\frac{22a}{15b}$. (12) $\frac{11(a-b)}{a+b}$. **2.** 1740 yds. per min. **3.** (1) $\frac{10}{3}$. (2) 200. (3) $54\frac{6}{11}$. (4) 144000. (5) 3784320000. (6) $\frac{17}{818400000}$. **4.** (1) 300. (2) $3\frac{1}{12}$. (3) $\frac{1}{9}$. (4) $\frac{1}{120}$. (5) $\frac{a}{b}$. (6) $\frac{a}{36b}$. (7) $\frac{a-b}{6(a+b)}$. **5.** $1\frac{2}{3}$ decimetres per second. **6.** The velocities are in the ratio of 9 to 10. **7.** (1) $\frac{22v}{15}$. (2) $\frac{88v}{3}$. (3) $192v$.

II. Page 16.

1. (1) 117000 ft. (2) 288000 ft. (3) 434450 ft. **2.** (1) 3520 min. (2) 1400 min. (3) 7924 min. **3.** 6 hours 31 min. nearly. **4.** $59\frac{7}{22}$ sec. **5.** 10 sec. **6.** 176 years, 343 days, 8 hours, taking a year to be 365 days. **7.** 97806 nearly.

III. Page 17.

1. $76\frac{4}{15}$. **2.** $56\frac{24}{31}$. **3.** $531\frac{5}{81}$. **4.** $15\frac{2027}{2221}$. **5.** 8 miles per hour.

IV. Page 23.

1. (1) 13; 67° 23′. (2) 17; 61° 56′. (3) 10; 60°. (4) 101; 78° 35′. (5) 13·86; 60°. (6) 14·49; 16° 4′. (7) 39; 32° 12′. (8) 37·6; 28° 37′. **2.** (1) $\sqrt{2a^2+2b^2}$. (2) $(m+n+1)^2+1$. (3) $\sqrt{2m^2+6n^2}$. (4) $\sqrt{4m^2+6n^2}$. (5) $\sqrt{1+6m^2}$. **3.** 3·6 knots per hour, in a direction making an angle of 33° 41′ with the east line. **4.** 6·18 ft. per sec. **5.** 1012·9 ft. per sec.

V. PAGE 30.

1. (1) $5\sqrt{3}, 5$. (2) $5\sqrt{2}, 5\sqrt{2}$. (3) $5, 5\sqrt{3}$. (4) $-5, 5\sqrt{3}$. (5) $-5\sqrt{3}, 5$. **2.** 3. 30°, 60°. **3.** (1) $\frac{u}{2}$; 60°. (2) $\frac{u\sqrt{3}}{2}$; 30°. (3) $\frac{u\sqrt{2}}{2}$; 45°. **4.** 14·999 miles per hour horizontally and ·15 mile per hour vertically. **5.** $20\sqrt{2} = 28·28$ ft. per sec. **6.** A velocity 6 in the direction of the given velocity 5. **7.** $10\sqrt{3} = 17·32$ ft. per sec. **8.** $60\sqrt{2} = 84·84$ days. **9.** (1) $6\frac{1}{2}$ miles or $3\frac{1}{2}$ miles according as the slower train is pursuing the faster, or the faster train pursuing the slower. (2) $2\frac{1}{2}$ or $12\frac{1}{2}$ m.

VI. PAGE 34.

1. $\frac{1}{20}$. **2.** $\frac{3}{10}$. **3.** Consistent with a uniform acceleration of $\frac{11}{30}$ ft. per sec. per sec. **4.** $\frac{11}{60}$. **5.** 6652800. **6.** (1) 29. (2) 39. (3) 19. (4) 11. **7.** (1) 38400. (2) 16. (3) $\frac{864000}{11}$. (4) $\frac{3981312000}{11}$. **8.** (1) 38880000. (2) 223948800000. (3) $\frac{54}{5}$. **9.** (1) $\frac{a}{b}$. (2) ac^2. (3) $\frac{ac^2}{b}$.

VII. PAGE 37.

1. 3600 ft. **2.** 24 miles per hour; $5(\sqrt{2}-1)$ min. **3.** 50. **4.** 5, 15, 25, 35, 45 ft. respectively. **5.** 38880000. **6.** 99 ft. **7.** 20, 45, 80, 125 in. respectively. **8.** $1466\frac{2}{3}$ ft. **10.** $p(\sqrt{2}-1)$ sec. **11.** As 2 : 9; as 2 : 3; as 1 : 9. **13.** (2) 10. (3) 22 centimetres per sec.; 80 centimetres per sec.

VIII. PAGE 42.

1. $\frac{15}{204·8}$; $\frac{81}{16}$. **2.** $(30·45)^3 \times 11·445$. **3.** (1) 120. (2) $6\frac{3}{10}$. (3) $\frac{176}{3} \times 20 \times 112 \times 200$. (4) $\frac{5}{4}$. **4.** (1) $\frac{1}{30}$. (2) 18200. (3) 100. (4) 6000000. **5.** $\frac{50 \times 112 \times 56}{3}$; $\frac{5 \times 112 \times 56}{3}$. **6.** 800000; 160000. **7.** $\frac{64}{9}$.

IX. PAGE 48.

1. 40. **2.** 18 dynes, $11\frac{1}{4}$ centimetres per sec. **3.** 160 dynes. **4.** $\frac{1}{82·95}$. **6.** $6\frac{2}{3}$. **7.** (1) $100 \times 20 \times 112 \times \frac{11}{9}$. (2) $100 \times 112 \times 20 \times \frac{121}{27}$. **8.** (1) 594 ft. (2) 6 ft. **9.** 30 ft. per sec. **10.** $\frac{2}{7}$. **11.** $200 \times 20 \times 112 \times 3$ poundals. **12.** Unit of force $= \frac{1}{192}$ poundal. **13.** Unit of force $= \frac{100 \times 1000}{60^2}$ dynes.

X. PAGE 54.

1. 1803·2 poundals; 56 lbs. weight. **2.** 40 poundals; 1·24 lbs. weight. **3.** ·8 ft. per sec. per sec. upwards. **4.** 36288 poundals; $16·2 \times 10 = 162$ poundals. **5.** As 5 : 32·2. **6.** 36·225 ft.; 24·15 ft. per sec. **7.** The acceleration would be doubled. **8.** As 15 : 109.

XI. PAGE 61.

2. Twice AD. **7.** 6 along OE. **8.** The three forces must act in the same straight line, the 7 and the 3 in the same direction, and the 10 in the opposite direction. **9.** $50\sqrt{2}$ lbs. **10.** 65 lbs. **11.** $10\sqrt{3}$ lbs.

XII. PAGE 63.

1. 60°. **2.** 109 lbs. **3.** 133 lbs. **4.** 115 lbs. **5.** 90°. **7.** A straight line bisecting at right angles the line representing the resultant. **8.** $P\sqrt{3}$ at right angles to P. **9.** (1) 180°. (2) 0°. **12.** 21·84 poundals.

XIII. PAGE 66.

1. (1) $10\sqrt{2}$. (2) 6. (3) $\sqrt{3}$. (4) $2\sqrt{3}$. (5) 96. (6) P. (7) P. **2.** (1) 135°. (2) 120°. (3) 150°. (4) 150°. (5) 44° 16′. (6) 120°. (7) 45°. **3.** $\sqrt{\{P^2 + Q^2 + (P + Q)^2(1 + 2\cos i) + 2PQ\cos 2i\}}$. **4.** P. **5.** $27 - 12\sqrt{2}$.

XIV. PAGE 70.

1. $\frac{25\sqrt{3}}{2} = 21·65$ lbs. and 12·5 lbs. **2.** $17\frac{1}{3}$ lbs. **3.** $13\frac{11}{38}$ lbs.

4. The tensions in the two parts of the string are equal, and each is equal to 57·73 lbs.

XV. Page 76.

1. (1) 20 units acting 14 inches from greater. (2) 6 units acting $46\frac{2}{3}$ inches from greater. **2.** 16 poundals along the line of one of the 8 poundals. **3.** 1 unit upwards 194 inches from A. **4.** 30 lbs. at A, 50 lbs. at B. **5.** 15 units 16 inches from force 1. **6.** 4 units $38\frac{1}{4}$ inches from A.

XVI. Page 82.

4. 15 at the point 6 inches from the end of the line. **5.** 4 units $27\frac{1}{4}$ inches from A. **6.** (1) 8 units at a point in AC $\frac{1}{8}a\sqrt{2}$ from the centre. (2) 18 units at the point in the line joining the centre of the hexagon with C distant $\frac{2a}{9}$ from the centre. (3) 2 units at the point in the diagonal produced through C distant $\frac{a\sqrt{2}}{2}$ from C.

XVII. Page 90.

1. 150. **2.** 35. **4.** $33Pa$. $\frac{5}{12}$. **5.** 31·6 nearly. 52. **6.** 13·11. 60.

XVIII. Page 100.

1. $\frac{32}{21}$ inches from centre of larger cube. **2.** $\sqrt{2}\times$ length of square. **3.** 98 ft. from the stern-post. **4.** $\frac{a}{9}$. **5.** $\frac{a}{12}$. **6.** $\frac{a\sqrt{3}}{18}$ inches from C.I. of the triangle. **7.** $\frac{12}{4\pi-\sqrt{3}}$ $=1\cdot1$ inches approximately from the centre of the circle. **8.** $\frac{1}{30}$ ft. from the centre of the square. **9.** $\frac{56}{39}, \frac{56}{42}, \frac{56}{45}$ ft. **12.** A circle.

XIX. Page 105.

1. 5 ft. 2 in. from the first-mentioned end. **2.** 15 lbs. **3.** $\frac{5}{32}\times$ length of rod. **4.** $\frac{7}{9}\times$ length of rod from larger end. **5.** $\frac{1}{6}\times$ length of rod from end. **7.** $\frac{6}{17}\times$ length of rod from end at which 16 oz. are hung. **8.** 60°. **9.** At a distance $\frac{c}{3}\,\frac{\cos 2\mathrm{B}}{\cos \mathrm{B}}$ from the right angle. **10.** Half the weight of the board. **12.** 2 oz. **13.** 50 lbs.; $50\sqrt{3}$ lbs. **14.** $\frac{1}{\sqrt{3}}\sqrt{b^2-a^2}$; $\frac{b\mathrm{W}\sqrt{3}}{2\sqrt{b^2-a^2}}$. **16.** Inclined 30° to the vertical. **17.** Thrusts along DA, DC, and CB; a tension in AB. Stress along DC = stress along AB $=\frac{60}{\sqrt{119}}$. Stress along AD = stress along BC $=\frac{144}{\sqrt{119}}$.

XX. PAGE 115.

1. 97·2 ft. **2.** 40 ft. per sec. 75 ft. **3.** (1) 461 ft. per sec. (2) 429 ft. per sec. (3) 3320·6 ft. (4) 28·8 sec. **4.** 5½. **5.** 640 ft. **6.** 32 ft. **7.** 509·7 centimetres. 2·04 sec. **8.** 1·799 sec. 1765 centimetres per sec. **9.** 5118·5 centimetres; 12539 centimetres. **10.** 224·25 ft. One 32 ft. per sec. upwards, the other 32 feet per sec. downwards. **13.** 432 ft. **14.** 980·33. **15.** 32·2. **16.** $(2+\sqrt{2})$ sec. $96+64\sqrt{2}$ ft. **17.** $h-\frac{1}{2}g(2n-1)$. **18.** $(v-\sqrt{v^2-2gh})\div g$. **19.** As 18 : 19.

XXI. PAGE 124.

1. 981·6. **2.** 31 lbs. **3.** 106704 dynes. 45 centimetres per sec. per sec. **4.** $\frac{2}{5}$ foot. 3⅕ ft. per sec. **5.** As 3 : 5. **6.** 30 ft. **7.** $\frac{5\sqrt{2}}{2}$ sec. **8.** $\frac{1}{9}g$. $\frac{1280}{9}$ poundals, $\frac{320}{3}$ poundals. 2·121 sec. **9.** 12½ ft. **10.** 30°. **11.** $\frac{135}{128}$ lbs. 5 ft. per sec. per sec. **13.** $32\sqrt{5}$ ft. per sec. ½.

XXII. PAGE 132.

1. (1) $\frac{1}{2\sqrt{3}}$. (2) $2-\sqrt{3}$. **2.** $\frac{11}{960}$. **3.** 12·23 sec.; 18348·6 centimetres. **5.** 4·52 sec.; 15·9 feet per sec.

XXIII. PAGE 141.

(*Take* $\pi^2=10$.)

1. 30 ft. per sec. **2.** $\frac{\pi}{6}$; $\frac{120}{\pi}$. **3.** 1327·5 lbs. **4.** As 1 : 2. **5.** (1) $18\frac{3}{4}$ lbs. (2) $187\frac{1}{2}$ lbs. **6.** 80 ft. per sec. **7.** 24 ft. per sec. **8.** As 210 : 1. **11.** 17 times faster. **12.** 1467·9.

XXIV. PAGE 146.

1. 514. **2.** (1) 39·128. (2) 39·15. (3) 39·1. (4) 39·014. **3.** (1) 978. (2) 982·03. (3) 983. **4.** 32·24. The pendulum would vibrate in $\frac{799}{800}$ second approximately. **5.** 32·13. **6.** 431 sec. nearly. **7.** 14·085 inches. **8.** It must be shortened $\frac{1}{2160}$ of its own length approximately. **9.** Lengthened ·6 per cent. **10.** It would lose $201\frac{1}{2}$ seconds per day. Shortened ·47 per cent.

XXV. PAGE 152.

1. (1) The moment of inertia of a circle about an axis through its centre perpendicular to its plane is $\frac{a^2}{2}$M; hence required moment of inertia is $\frac{3}{2}a^2$M. (2) The square may be considered as a rectangular block. $\frac{2}{3}$Ma^2. (3) 2Ma^2. (4) $\frac{16}{3}$Ma^2, where $2a$ is the length of the rod. **2.** 2·69 feet from the end. **3.** 1·08 inches. ·03. **4.** The angle whose tangent is $\frac{121}{216}$. **5.** 9·8.

XXVI. PAGE 162.

1. 240000 foot-pounds. $68571\frac{3}{7}$ lbs. $145\frac{5}{11}$. **2.** 67570. **3.** 1612800. **4.** $13\frac{1}{3}$ ft. per sec. **5.** 15000. **7.** 3906250 foot-pounds. 46875000 lbs. **8.** (1) The momenta are equal. The kinetic energies are as $m' : m$. (2) The momenta are as $\sqrt{m} : \sqrt{m'}$. The kinetic energies are equal. **9.** 1341·6 ft. per sec. **10.** $89\frac{3}{5}$. **11.** 550 lbs. **12.** $85\frac{1}{3}$. **13.** 128.

14. 2143 nearly. **15.** 250. 129·9. **16.** $\sqrt{2U \div m}$. $13\frac{1}{3}$ ft. per sec. **17.** 109·375.

XXVII. PAGE 172.

1. $1365\frac{1}{3}$ ft. per sec. **2.** (1) $15\frac{15}{16}$ ft. per sec. (2) 0. **3.** $-\frac{2197}{864}, \frac{2171}{864}$ ft. per sec.; 95·2 foot-poundals nearly. **5.** $m m'(u-u')^2 \div 2(m+m')$. **7.** $20\frac{5}{8}$, $13\frac{1}{8}$ centimetres per sec. 9228·6. **8.** $2n-1:1$. **9.** $\frac{3}{4}$. $41\frac{1}{4}$ ft. after the first impact. **10** $\sqrt{K \div h}$. **11**· Eight-ninths.

XXVIII. PAGE 190.

1. 32·96. 26·96. **2.** ·67. **3.** 40 lbs. **4.** 41 lbs. **5.** 29·16 grammes. As 127 : 125. **6.** 4 lbs. at A, 6 lbs. at B. **7.** $1\frac{3}{5}$ ft. from fulcrum. **10.** 40 lbs., 46·19 lbs. **13.** 7543 lbs., taking $\pi = \frac{22}{7}$. **14.** 3394·3. **15.** 5·923 lbs. Acceleration of power is ·154 g., of weight is ·077 g. **16.** $\frac{49}{550}$ lbs. **17.** Half the weight of the man. **18.** $83\frac{1}{3}$ lbs. **19.** $W(\sqrt{3}-1) \div 2\sqrt{3}$.

XXIX. PAGE 206.

1. 656·25 lbs. **2.** 36750. **3.** 27000000, 13500000 grammes. **4.** 9 ft. **5.** 24000, 300000 lbs. **6.** 25000 lbs. **7.** As 1 : 3. **9.** 4200 lbs. **10.** 16 ft., 4 ft. **11.** 1500 lbs. **12.** 60°. The depth of water in the first position must not be less than $\sqrt{3} \times$ radius of base. **13.** 2604·17, 5208·34 lbs. **15.** $3\sqrt{3}$ inches. **16.** 40 grammes. **17.** If h denote the perpendicular from the vertex to the base, the lines are at distances $h \div \sqrt[3]{3}$ and $h\sqrt[3]{2} \div \sqrt[3]{3}$ from the vertex. **18.** 50 inches, $10\sqrt{5}(\sqrt{3}-\sqrt{2})$ inches. **19.** $AP = \frac{3}{5} AB$.

XXX. PAGE 223.

1. 11200. **2.** 9800. **3.** 48 to 1. **4.** 64 cubic centimetres. 4. **5.** $\frac{10}{9}$. **6.** 1·5. **7.** 261800 lbs., 9350 lbs. **8.** ·88. **9.** 327·6, 540, 819, 981 lbs. **10.** 2354·7 lbs. **11.** (1) $s : s'$. (2) Equal volumes. **12.** As 1075 : 1825 : 3139. **13.** 195720, 156352·3, 109446·6 grains. **14.** 1·85 ft. on one side, 1·15 ft. on the other. **15.** $7\frac{1}{5}$ lbs. **16.** As AC : BC cosBCA. **18.** $\frac{19}{21}$ oz. $30\frac{2}{21}$ oz. **19.** $\frac{5}{12}$ sq. centimetre. **20.** $\frac{70}{81}$ sq. inch. **21.** ·8 centimetre. **22.** 20

cubic centimetres. 6·4. **23.** $\frac{8}{9}$ lbs., $\frac{10}{11}$ lbs. **24.** 1·1, $\cdot\dot{4}\dot{3}$. **25.** 354·3 lbs. **26.** 7 inches. $\frac{6}{7}$. **27.** It would float. **28.** $\frac{60}{209}$. **29.** $\frac{2}{3}$, 6. **31.** It will neither rise nor fall. **32.** $2\frac{2}{3}$. **33.** 161·94 lbs. 2·59. **34.** $\frac{18}{5}$. **35.** $\frac{4}{5}$. **36.** $\frac{1}{5}$. **37.** 69·4. **38.** $\frac{1}{4}$. **39.** $\frac{14}{17}$. **40.** 750 oz. **41.** $\frac{36}{95}$ cubic ft. **42.** $(b-as)\div(1-s)$ grammes. $(b-a)\div(1-s)$ cubic centimetres. **43.** $\frac{5}{6}$. **44.** 175 grains. **45.** 10·62 oz. **46.** 25 grammes. **47.** $W_0-W_1:W_0-W_2:W_0-W_3:\ldots$ **48.** $\frac{18}{19}$. **49.** The point $2\frac{7}{19}$ inches above the point which is in the surface when it floats in water.

XXXI. PAGE 241.

1. 240 million ergs. **2.** 38·9 ft. **3.** A pressure of $(h-8\frac{1}{2})$ ft. of water, where h is the height of the water barometer. **4.** 41130 lbs. **5.** $\{(a-h)(H-h)+h'(a-h')\}\div(a-h')$. **6.** 74·8 centimetres. **7.** In the tube a pressure of 25 inches of mercury, in receiver a pressure of 21 inches. **8.** $3\frac{1}{5}$ atmospheres. **9.** $(\frac{1}{6}h-\frac{1}{2})$ inches, h being height of barometer. **10.** 3 inches very nearly. **11.** $\frac{14}{15}$ gramme. **12.** $\frac{3}{7}$ ft. **13.** $\frac{7}{8}$ cubic inch. **14.** $\frac{7}{12}\times$ length of tube. **15.** $\frac{1}{10}$. **16.** 3 inches. **18.** 138·89. **19.** 160. **20.** 2·15. **21.** $\frac{19}{100}$ atmosphere. **22.** $8\frac{8}{9}$. **23.** 20·4 ft. **24.** 537·6 millimetres.

XXXII. PAGE 248.

1. (1) 21·9. (2) 30·3. (3) 40·7. (4) 50·2. (5) 9·1. (6) 20·8. (7) 37·7. **2.** (1) 10·6. (2) 17·6. (3) 29·9. (4) 34·7. (5) 101. (6) 7. (7) 20·2. (8) 9·5. (9) 20. (10) 22·8. (11) 28·9. (12) 43·2. (13) 9·4. **3.** (1) 9·1. (2) 21·3. (3) 34. (4) 73·6. (5) 79·8. (6) 39·1. (7) 2·1. (8) 5·1. **5.** The results are in lbs.—(1) 6·6, 50·4. (2) 13·4, 51·8. (3) 20·7, 54·1. (4) 28·9, 57·7. (5) 50, 70·7. (6) 86·6, 100. (7) 120·7, 130·7. (8) 186·6, 193·2. (9) 379·8, 383·1.

EXAMINATION PAPERS.

I. FIRST STAGE.—PAGE 272.

1. (*c*) In the axis one-third of the length of the axis from the base. **3.** The forces are the weight of the rod through its middle point, the reaction of the wall at B acting horizontally, and the reaction of the hinge. These forces must meet in a point. **4.** 12 lbs.; 18 lbs. **5.** The sensibility of a balance is its capability of detecting minute differences of the masses placed in the scale-pans. **6.** 12000; 5000. **7.** 1920 ft. per sec. 38400. **8.** 88 ft. per sec. $5\frac{1}{2}$. **9.** $15\frac{5}{8}$ foot-pounds. $31\frac{1}{4}$ ft. **10.** 75 lbs. **11.** $\frac{61}{40}$. **12.** 875.

II.

SECOND STAGE.—PAGE 273.

1. 26·6 units. **4.** Reaction at C acts horizontally; the reaction at B in a direction making with the vertical an angle whose tangent is $\frac{\sqrt{3}}{2}$. Reaction at B : reaction at C : given weight $= \sqrt{7} : \sqrt{3} : 2$. **5.** The angle whose tangent is ·31. **7.** 860000; 20640000. **8.** $\frac{1600}{9}$ ft. per sec. **9.** $\frac{15}{4}$, $\frac{7}{2}$ ft. per sec. **10.** 55561 foot-pounds. Take $\pi = \frac{22}{7}$. 44 turns. **11.** At a point distant $\frac{5}{9} \times$ height of rectangle from upper edge. **12.** (*a*) $\frac{19}{20}$. (*b*) 1.

III. UNIVERSITY OF LONDON.

MATRICULATION EXAMINATION.—PAGE 275.

A. **1.** 56. **2.** 15·9375 oz. **3.** $P = W$. **4.** $\frac{1}{5} \times$ side of square. **5.** 2400 foot-pounds. 50·1 ft. per sec.

B. **6.** 2500, 3500, 3000 oz. **7.** $\frac{12}{11}$. **8.** The lift-pump is a modification of common pump. The piston, as in forcing pump, works air-tight through a cover, and there is a valve placed at H (see figure, Art. 163) as in forcing pump. A pipe leads from H up to a cistern. For contrivance referred to see books on Practical Mechanics.

IV. UNIVERSITY OF EDINBURGH.

MEDICAL PRELIMINARY EXAMINATION.—PAGE 277.

2. 18 yds. per min. **3.** See Art. 106. **4.** 3 ft. per sec. per sec. $13\frac{1}{2}$ ft. per sec. $67\frac{1}{2}$ ft. **5.** 64 ft. **7.** 22 lbs. **8.** No. Three of the forces pass through a point, and the remaining force does not. **10.** 3410 lbs. per sq. ft. taking the atmospheric pressure at 15 lbs. per sq. inch.

V. UNIVERSITY OF EDINBURGH.

M.A. Pass Examination.—Page 278.

3. $\sqrt{mg \div 6}$ ft. per sec. **5.** (*b*) The C. G. of the triangle. (*c*) The centre of the circle inscribed in the triangle. **8.** (1) A vertical straight line. (2) A circle. **9.** When the area is horizontal. Divide the side which is not horizontal into three parts which are as $1 : \sqrt{2}-1 : \sqrt{3}-\sqrt{2}$. Horizontal lines through the points of division are the lines required. **11.** As $p+q : q$.

VI. UNIVERSITY OF GLASGOW.

M.A. Pass Examination.—First Paper.—Page 279.

3. 60°. $2\sqrt{19}$. **4.** At its middle point. **6.** The second system of Art. 126 reversed, the weight W becoming the beam and the beam becoming the weight. More advantageous to have pulleys heavy. **7.** $\frac{g}{6}$. **8.** $\frac{1}{8}m(u+v)^2$. **10.** 14.8 lbs., taking $\pi = \frac{22}{7}$. **11.** $\frac{484}{9}m$ foot-pounds, where m is the mass of the train in pounds.

VII.

Second Paper.—Page 281.

2. 2906·25 lbs. **4.** 799100 centimetres approximately.

VIII. WOOLWICH ROYAL ACADEMY.

Statics.—Page 281.

3. Components must be equally inclined to resultant. The problem thus reduces to a simple geometrical problem. **6.** 1 lb. downwards 5 ft. from first force. **8.** $\frac{4}{15}$ nearly. **9.** In the line drawn from the first position of the C.G. perpendicular to the base, at a distance of one-twelfth of this line from first position of C.G. **10.** Use theorem of preceding question. **11.** The principle of virtual velocities is the principle of conservation of energy under another (and older) name.

IX. DYNAMICS.*

1. In a direction making with the bank an angle whose cosine is $\frac{2}{3}$. **2.** $\frac{v\sqrt{3}}{2}$. **6.** The formula for the range is $u^2\sin 2\alpha \div g$. This is greatest when $\alpha = 45°$, and its value is then $u^2 \div g$. $7\frac{31}{192}$ miles. In practice the range would be much less than this owing to the resistance of the atmosphere. **8.** The second projectile would appear to rise vertically with uniform velocity. **9.** This question involves results obtained in Conic Sections.

X.

SOLIDS. PAGE 284.

1. The angular velocity of the moving point is equal to its linear velocity divided by its distance only when the point is moving perpendicular to the line joining the two points. In other cases the angular velocity is equal to the component of its linear velocity perpendicular to this line divided by its distance. $10 \cos 30° \div 12, = 5\sqrt{3}/12$ radians per second. **3.** Produce DC to E, making DE equal to $\frac{3}{2} \times$ side of square; then the resultant is a force of 8 units, acting at E, parallel to and in the same direction as the first force. If the first force acted from D to A, the resultant would be a force of 8 units acting parallel to and in the same direction as the first force through a point F on CD produced, such that DF is $\frac{3}{2} \times$ side of square. **5.** If W is the weight of the particle, and i the inclination of the plane, the friction called into play is the force required to balance the component of the weight down the plane, and is therefore equal to $W \sin i$. The particle will stay at rest, when the inclination is increased, as long as $W \sin i$ is less than $\mu \times$ normal pressure, that is, less than $\mu \times W \cos i$, where μ is the coefficient of friction. Hence the particle will stay at rest as long as $\sin i$ is less than $\mu \cos i$; that is, as long as $\tan i$ is less than μ; that is, as long as $\tan i$

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* The word *Dynamics* is here used for the word *Kinetics.*

is less than $\tan\phi$, where ϕ is the angle of friction. Hence the particle will stay at rest if i, the inclination, is less than ϕ, the angle of friction. **6.** If a is the height of C above plane, i the inclination of the rod to the horizon, and μ the coefficient of friction, the length of the rod is equal to $2a\mu \div \sin i \cos i \, (\mu \cos i + \sin i)$. The steepest inclination is the angle whose tangent is $1 \div \mu$. **9.** The time down PQ is $\sqrt{2(d+d')/g}$, where d is the diameter of the upper circle, d' the diameter of the lower circle, and g the acceleration of gravity. The times in which PA and AQ are described are in the ratio $\sqrt{d} : \sqrt{(d+d')} - \sqrt{d}$. **10.** The velocity would be retarded if the inclination of the inclined plane were less than the angle of friction. $625(\sqrt{18}+\sqrt{6})/8 = 523$ feet nearly. **12.** 9 feet per second. Five-eighths of the energy of the system disappears when the thread is drawn tight.

XI.—UNIVERSITY OF LONDON.

INTERMEDIATE B.SC. PAGE 287.

1. The fixed point is the point C (say), which divides AB so that $AC : BC = n : m$. **2.** The pressure on each wire is $(M+2m)/\sqrt{2}$. **3.** Let ABC be the triangle. The resultant of the forces at B and C acts at the point D (say), where the bisector of the angle A meets BC (Euclid vi. 3); and the resultant of the three forces at A, B, C acts at some point in AD. Similarly the resultant acts at a point lying on each of the bisectors of the angles B and C, and the three bisectors meet in the centre of the inscribed circle. **4.** The centre of gravity of the cardboard, when folded, lies in the line joining the centre of the square to the angular point opposite the corner which is bent over, and at a distance from the centre equal to $a\sqrt{2}/48$ inches. **5.** The part of the steel-yard on the side of the fulcrum opposite to that on which the scale-pan is, will be marked 1, 2, 3... lbs. at distances of 1, 4, 7... inches respectively from the fulcrum. The greatest weight which the yard can measure is $11\frac{2}{3}$ lbs. **6.** 1760 feet per second. **7.** $66\frac{2}{3}$ feet per second. $4\frac{1}{6}$ seconds. **9.** ·002205. **10.** If $m \sin\alpha < m'$, the mass m will move *up* the plane with the acceleration $g(m' - m \sin\alpha)/(m + m')$. If $m \sin\alpha > m'$, the mass m will move *down* the plane with the acceleration $g(m \sin\alpha - m')/(m + m')$. If $m \sin\alpha = m'$, the system will remain at rest. In all the cases the tension of the string is $mm'g(1 + \sin\alpha)/(m + m')$ poundals, m and m' being expressed in lbs.

XII.

SOLIDS. PAGE 289.

1. A moving point will have no angular velocity with reference to a fixed point when the first point is moving in the straight line joining the two points. If v denotes Q's constant velocity, c the perpendicular distance from P to AB, and r the distance PQ, the angular velocity of Q with respect to P is cv/r^2. **3.** The angle between the forces 12 and 15 is 91°, that between the forces 15 and 19 is 141°, and that between the forces 12 and 19 is 128°. **4.** The co-ordinates of the centre are $(5, -6)$. **5.** Taking A to be above B, the rod AC is in tension and the rod BC in compression. The weight W is supported by the forces acting along AC and BC respectively, and the three forces are proportional to the sides of the triangle ABC. In the numerical case the stress in AC is 9 tons, and the stress in BC is $13\frac{1}{2}$ tons. **6.** Co-efficient of friction is $\frac{1}{2}$. The inclination when the friction is one-half the limiting friction is the angle whose tangent is 2. **7.** 7740 foot-pounds. **8.** Draw a straight line ABX, and take the point B such that AB contains as many units on any convenient scale as the length of the rope contains feet. Draw AD and BC, perpendicular to AB, each containing as many units as the weight at the end of the rope contains pounds. Then the area of the rectangle ABCD represents the work done in lifting the *weight alone.* Produce AD to a point E such that DE contains as many units as the weight of the rope contains pounds, and join EC. Then the area of the triangle CDE represents the work done in lifting the *rope alone.* The total work done is therefore represented by the sum of those two areas, that is, by the area of the trapezium ABCE. **9.** At a distance from N, measured horizontally, equal to $2\sqrt{3} \times \mathrm{AS}$. 60°. **10.** If l denotes the length of the rope in feet, x the length hanging over the edge, m the mass of the thread in pounds, and g the acceleration of gravity in feet and seconds—then the results are (*a*) gx/l ft./sec.^{2} (*b*) $mx^2/2l$ foot-pounds. (*c*) $\sqrt{gx^2/l}$ ft./sec. **11.** If h denotes the given height and e the coefficient of restitution, the heights of the first and second rebounds are e^2h and e^4h respectively.

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Intermediate B.Sc. Pages 291–293.

1. Thrusts in CA and CB are 80 lbs. and 60 lbs. respectively; the tension of the cord is 48 lbs.; the fractions of the load sustained at A and B by the plane are $\frac{16}{25}$ and $\frac{9}{25}$ respectively, that is, 64 lbs. and 36 lbs respectively. **2.** In the first part of the question, the resultant is a force of magnitude BC, acting parallel to BC through the point D in AB produced such that BD is equal to AB. In the second part of the question, the resultant is zero, or the forces are in equilibrium. **3.** The loads at the angular points opposite the sides 5, 4, 3 must be $8\frac{1}{3}$, $6\frac{2}{3}$, 5 lbs. respectively. **4.** 2·12 inches. In the second part of the question, the "certain given angle" is twice the angle ϕ, which is such that $\cos\phi = \mathrm{R}/2\mathrm{P}$. **5.** A force of 2·12 lbs. **6.** 10650 feet. **7.** $32{\cdot}2 \times 12 \times 60^4 \div (39{\cdot}37 \times 1000), = 127196{\cdot}96$. **8.** $\frac{20}{101}$ foot a second. **9.** $12\frac{2}{9}$ lbs. The acceleration with which the mass m falls is $g(m-2M) \div (m+2M)$, where g represents the acceleration of gravity. **10.** (i) 860 seconds, (ii) 476 seconds, taking g equal to 32 in ft.-sec. units.

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